

# Weekly Homework Template Tutorial

Your Name  
Course-Semester

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**Problem 6.** a) Suppose an entire function  $f$  is bounded by  $M$  along  $|z| = R$ . Show that the coefficients  $C_k$  in its power series expansion about 0 satisfy

$$|C_k| \leq \frac{M}{R^k}.$$

b) Suppose a polynomial is bounded by 1 in the unit disc. Show that all its coefficients are bounded by 1.

**Solution** Part a): Since  $f$  is an entire function it can be expressed as an infinite power series, i.e.

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = \sum_{k=0}^{\infty} C_k z^k.$$

If we recall Cauchy's Integral we have

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} dw,$$

carefully notice that  $\frac{1}{w-z} = \frac{1}{w} \cdot \frac{1}{1-\frac{z}{w}}$  can be written as a geometric series. We have

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} dw &= \frac{1}{2\pi i} \int_{\gamma} \left\{ \frac{f(w)}{w} \cdot \left( \frac{1}{1-\frac{z}{w}} \right) \right\} dw \\ &= \frac{1}{2\pi i} \int_{\gamma} \left\{ \frac{f(w)}{w} \cdot \left( 1 + \frac{z}{w} + \frac{z^2}{w^2} + \frac{z^3}{w^3} + \dots \right) \right\} dw \\ &= \left( \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w} dw \right) z^0 + \left( \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w^2} dw \right) z^1 + \left( \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w^3} dw \right) z^2 \dots \end{aligned}$$

Now take the modulus of  $C_k$  to get

$$|C_k| = \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w^{k+1}} dw \right| \leq \frac{1}{2\pi} \int_{\gamma} \frac{|f(w)|}{|w^{k+1}|} |dw| \leq \frac{M}{2\pi} \int_{\gamma} \frac{|dw|}{|w^{k+1}|}$$

Then integrate along  $\gamma(\theta) = Re^{i\theta}$  for  $\theta \in [0, 2\pi]$  to get

$$|C_k| \leq \frac{M}{2\pi} \int_0^{2\pi} \frac{|iRe^{i\theta} d\theta|}{|R^{k+1}e^{ik\theta}|} = \frac{M}{2\pi \cdot R^k} \int_0^{2\pi} d\theta = \frac{M}{R^k}.$$

Hence,  $|C_k| \leq \frac{M}{R^k}$ .