

1  
2  
3  
4  
5  
6  
7  
8  
9  
10  
11  
12  
13  
14  
15  
16  
17  
18  
19  
20  
21  
22  
23  
24  
25  
26  
27  
28  
29  
30  
31  
32  
33  
34  
35  
36  
37  
38  
39  
40  
41  
42

# Sample Paper for the aomart Class

By AMERICAN MATHEMATICAL SOCIETY and BORIS VEYTSMAN

## Abstract

This is a test file for `aomart` class based on the `testmath.tex` file from the `amsmath` distribution.

It was changed to test the features of the Annals of Mathematics class.

## Contents

1. Introduction	18
2. Enumeration of Hamiltonian paths in a graph	18
3. Main theorem	19
4. Application	22
5. Secret key exchanges	23
6. Review	23
7. One-way complexity	28
8. Various font features of the <code>amsmath</code> package	35
8.1. Bold versions of special symbols	35
8.2. “Poor man’s bold”	35
9. Compound symbols and other features	36
9.1. Multiple integral signs	36
9.2. Over and under arrows	36
9.3. Dots	36
9.4. Accents in math	37
9.5. Dot accents	37
9.6. Roots	37
9.7. Boxed formulas	37
9.8. Extensible arrows	38
9.9. <code>\overset</code> , <code>\underset</code> , and <code>\sideset</code>	38
9.10. The <code>\text</code> command	38
9.11. Operator names	38

---

The class was commissioned by Annals of Mathematics.  
© 2008–2012 Boris Veytsman.

<u>1</u>	9.12. <code>\mod</code> and its relatives	39
<u>2</u>	9.13. Fractions and related constructions	39
<u>3</u>	9.14. Continued fractions	41
<u>4</u>	9.15. Smash	41
<u>5</u>	9.16. The ‘cases’ environment	41
<u>6</u>	9.17. Matrix	42
<u>7</u>	9.18. The <code>\substack</code> command	43
<u>8</u>	9.19. Big-g-g delimiters	44
<u>9</u>	References	44

101112

## 1. Introduction

13

14 This paper demonstrates the use of `aomart` class. It is based on `testmath.tex`  
15 from  $\mathcal{A}\mathcal{M}\mathcal{S}$ -L $\text{\TeX}$  distribution. The text is (slightly) reformatted according to  
16 the requirements of the `aomart` style. See also [LO74, Zarh92, MO08, Arn89,  
17 Mic48, Mic38, Zarb, Zara, dGWH+92].

18 It is always a pleasure to cite Knuth [Knu94].

19

## 2. Enumeration of Hamiltonian paths in a graph

20

21 Let  $\mathbf{A} = (a_{ij})$  be the adjacency matrix of graph  $G$ . The corresponding  
22 Kirchhoff matrix  $\mathbf{K} = (k_{ij})$  is obtained from  $\mathbf{A}$  by replacing in  $-\mathbf{A}$  each  
23 diagonal entry by the degree of its corresponding vertex; i.e., the  $i$ th diagonal  
24 entry is identified with the degree of the  $i$ th vertex. It is well known that

$$\text{(1)} \quad \det \mathbf{K}(i|i) = \text{the number of spanning trees of } G, \quad i = 1, \dots, n$$

27 where  $\mathbf{K}(i|i)$  is the  $i$ th principal submatrix of  $\mathbf{K}$ .

28 `\det\mathbf{K}(i|i)=\text{the number of spanning trees of }G`,

29

30 Let  $C_{i(j)}$  be the set of graphs obtained from  $G$  by attaching edge  $(v_i v_j)$   
31 to each spanning tree of  $G$ . Denote by  $C_i = \bigcup_j C_{i(j)}$ . It is obvious that the  
32 collection of Hamiltonian cycles is a subset of  $C_i$ . Note that the cardinality of  
33  $C_i$  is  $k_{ii} \det \mathbf{K}(i|i)$ . Let  $\widehat{X} = \{\hat{x}_1, \dots, \hat{x}_n\}$ .

34 `\wh X=\{\hat{x}_1, \dots, \hat{x}_n\}`

35 Define multiplication for the elements of  $\widehat{X}$  by

$$\text{(2)} \quad \hat{x}_i \hat{x}_j = \hat{x}_j \hat{x}_i, \quad \hat{x}_i^2 = 0, \quad i, j = 1, \dots, n.$$

38 Let  $\hat{k}_{ij} = k_{ij} \hat{x}_j$  and  $\hat{k}_{ij} = -\sum_{j \neq i} \hat{k}_{ij}$ . Then the number of Hamiltonian cycles  
39  $H_c$  is given by the relation [LC84]

$$\text{(3)} \quad \left( \prod_{j=1}^n \hat{x}_j \right) H_c = \frac{1}{2} \hat{k}_{ij} \det \widehat{\mathbf{K}}(i|i), \quad i = 1, \dots, n.$$

42

Are these  
quotations  
necessary?

1 The task here is to express (3) in a form free of any  $\hat{x}_i, i = 1, \dots, n$ . The result  
2 also leads to the resolution of enumeration of Hamiltonian paths in a graph.

3 It is well known that the enumeration of Hamiltonian cycles and paths  
4 in a complete graph  $K_n$  and in a complete bipartite graph  $K_{n_1 n_2}$  can only be  
5 found from *first combinatorial principles* [HP73]. One wonders if there exists a  
6 formula which can be used very efficiently to produce  $K_n$  and  $K_{n_1 n_2}$ . Recently,  
7 using Lagrangian methods, Goulden and Jackson have shown that  $H_c$  can be  
8 expressed in terms of the determinant and permanent of the adjacency matrix  
9 [GJ81]. However, the formula of Goulden and Jackson determines neither  $K_n$   
10 nor  $K_{n_1 n_2}$  effectively. In this paper, using an algebraic method, we parametrize  
11 the adjacency matrix. The resulting formula also involves the determinant  
12 and permanent, but it can easily be applied to  $K_n$  and  $K_{n_1 n_2}$ . In addition,  
13 we eliminate the permanent from  $H_c$  and show that  $H_c$  can be represented by  
14 a determinantal function of multivariables, each variable with domain  $\{0, 1\}$ .  
15 Furthermore, we show that  $H_c$  can be written by number of spanning trees of  
16 subgraphs. Finally, we apply the formulas to a complete multigraph  $K_{n_1 \dots n_p}$ .

17 The conditions  $a_{ij} = a_{ji}, i, j = 1, \dots, n$ , are not required in this paper.  
18 All formulas can be extended to a digraph simply by multiplying  $H_c$  by 2.  
19 Some other discussion can be found in [Fre08, Fre94].  
20

### 21 3. Main theorem

22

23 *Notation.* For  $p, q \in P$  and  $n \in \omega$  we write  $(q, n) \leq (p, n)$  if  $q \leq p$  and  
24  $A_{q,n} = A_{p,n}$ .

25 `\begin{notation}` For  $p, q \in P$  and  $n \in \omega$   
26 `\dots`  
27 `\end{notation}`  
28

29 Let  $\mathbf{B} = (b_{ij})$  be an  $n \times n$  matrix. Let  $\mathbf{n} = \{1, \dots, n\}$ . Using the properties  
30 of (2), it is readily seen that

31 LEMMA 3.1.

$$\text{(4)} \quad \prod_{i \in \mathbf{n}} \left( \sum_{j \in \mathbf{n}} b_{ij} \hat{x}_i \right) = \left( \prod_{i \in \mathbf{n}} \hat{x}_i \right) \text{per } \mathbf{B}$$

32 where  $\text{per } \mathbf{B}$  is the permanent of  $\mathbf{B}$ .

33 Let  $\widehat{Y} = \{\hat{y}_1, \dots, \hat{y}_n\}$ . Define multiplication for the elements of  $\widehat{Y}$  by

$$\text{(5)} \quad \hat{y}_i \hat{y}_j + \hat{y}_j \hat{y}_i = 0, \quad i, j = 1, \dots, n.$$

34 Then, it follows that  
35  
36  
37  
38  
39  
40  
41  
42

1 LEMMA 3.2.

2  
3 (6) 
$$\prod_{i \in \mathbf{n}} \left( \sum_{j \in \mathbf{n}} b_{ij} \hat{y}_j \right) = \left( \prod_{i \in \mathbf{n}} \hat{y}_i \right) \det \mathbf{B}.$$

4  
5 Note that all basic properties of determinants are direct consequences of  
6 Lemma 3.2. Write

7  
8 (7) 
$$\sum_{j \in \mathbf{n}} b_{ij} \hat{y}_j = \sum_{j \in \mathbf{n}} b_{ij}^{(\lambda)} \hat{y}_j + (b_{ii} - \lambda_i) \hat{y}_i \hat{y}_j$$

9  
10 where

11  
12 (8) 
$$b_{ii}^{(\lambda)} = \lambda_i, \quad b_{ij}^{(\lambda)} = b_{ij}, \quad i \neq j.$$

13 Let  $\mathbf{B}^{(\lambda)} = (b_{ij}^{(\lambda)})$ . By (6) and (7), it is straightforward to show the following  
14 result:

15  
16 THEOREM 3.3.

17  
18 (9) 
$$\det \mathbf{B} = \sum_{l=0}^n \sum_{I_l \subseteq \mathbf{n}} \prod_{i \in I_l} (b_{ii} - \lambda_i) \det \mathbf{B}^{(\lambda)}(I_l | I_l),$$

19  
20 where  $I_l = \{i_1, \dots, i_l\}$  and  $\mathbf{B}^{(\lambda)}(I_l | I_l)$  is the principal submatrix (obtained from  
21  $\mathbf{B}^{(\lambda)}$  by deleting its  $i_1, \dots, i_l$  rows and columns).

22  
23 Remark 3.1 (convention). Let  $\mathbf{M}$  be an  $n \times n$  matrix. The convention  
24  $\mathbf{M}(\mathbf{n} | \mathbf{n}) = 1$  has been used in (9) and hereafter.

25 Before proceeding with our discussion, we pause to note that Theorem 3.3  
26 yields immediately a fundamental formula which can be used to compute the  
27 coefficients of a characteristic polynomial [MM64]:

28  
29 COROLLARY 3.4. Write  $\det(\mathbf{B} - x\mathbf{I}) = \sum_{l=0}^n (-1)^l b_l x^l$ . Then

30  
31 (10) 
$$b_l = \sum_{I_l \subseteq \mathbf{n}} \det \mathbf{B}(I_l | I_l).$$

32  
33 Let

34  
35 (11) 
$$\mathbf{K}(t, t_1, \dots, t_n) = \begin{pmatrix} D_1 t & -a_{12} t_2 & \dots & -a_{1n} t_n \\ -a_{21} t_1 & D_2 t & \dots & -a_{2n} t_n \\ \dots & \dots & \dots & \dots \\ -a_{n1} t_1 & -a_{n2} t_2 & \dots & D_n t \end{pmatrix},$$

36  
37  
38 
$$\begin{pmatrix} D_1 t & -a_{12} t_2 & \dots & -a_{1n} t_n \\ -a_{21} t_1 & D_2 t & \dots & -a_{2n} t_n \\ \dots & \dots & \dots & \dots \\ -a_{n1} t_1 & -a_{n2} t_2 & \dots & D_n t \end{pmatrix}$$
  
39  
40  
41  
42

$\frac{1}{2}$  where

$$\frac{3}{4} \quad (12) \quad D_i = \sum_{j \in \mathbf{n}} a_{ij} t_j, \quad i = 1, \dots, n.$$

$\frac{5}{6}$  Set

$$\frac{7}{8} \quad D(t_1, \dots, t_n) = \frac{\delta}{\delta t} \det \mathbf{K}(t, t_1, \dots, t_n)|_{t=1}.$$

$\frac{9}{10}$  Then

$$(13) \quad D(t_1, \dots, t_n) = \sum_{i \in \mathbf{n}} D_i \det \mathbf{K}(t = 1, t_1, \dots, t_n; i|i),$$

$\frac{11}{12}$  where  $\mathbf{K}(t = 1, t_1, \dots, t_n; i|i)$  is the  $i$ th principal submatrix of  $\mathbf{K}(t = 1, t_1, \dots, t_n)$ .

$\frac{13}{14}$  Theorem 3.3 leads to

$$(14) \quad \det \mathbf{K}(t_1, t_1, \dots, t_n) = \sum_{I \in \mathbf{n}} (-1)^{|I|} t^{n-|I|} \prod_{i \in I} t_i \prod_{j \in I} (D_j + \lambda_j t_j) \det \mathbf{A}^{(\lambda t)}(\bar{I}|\bar{I}).$$

$\frac{16}{17}$  Note that

$$(15) \quad \det \mathbf{K}(t = 1, t_1, \dots, t_n) = \sum_{I \in \mathbf{n}} (-1)^{|I|} \prod_{i \in I} t_i \prod_{j \in I} (D_j + \lambda_j t_j) \det \mathbf{A}^{(\lambda)}(\bar{I}|\bar{I}) = 0.$$

$\frac{20}{21}$  Let  $t_i = \hat{x}_i, i = 1, \dots, n$ . Lemma 3.1 yields

$$(16) \quad \left( \sum_{i \in \mathbf{n}} a_{li} x_i \right) \det \mathbf{K}(t = 1, x_1, \dots, x_n; l|l) \\ = \left( \prod_{i \in \mathbf{n}} \hat{x}_i \right) \sum_{I \subseteq \mathbf{n} - \{l\}} (-1)^{|I|} \text{per } \mathbf{A}^{(\lambda)}(I|I) \det \mathbf{A}^{(\lambda)}(\bar{I} \cup \{l\} | \bar{I} \cup \{l\}).$$

$\frac{27}{28}$   $\begin{array}{l} \backslash \text{begin}\{\text{multline}\} \\ \backslash \text{biggl}(\sum_{i \in \mathbf{n}} a_{li} x_i) \backslash \text{biggr} \\ \backslash \det \mathbf{K}(t=1, x_1, \dots, x_n; l | l) \\ = \backslash \text{biggl}(\prod_{i \in \mathbf{n}} \hat{x}_i) \backslash \text{biggr} \\ \backslash \sum_{I \subseteq \mathbf{n} - \{l\}} \\ (-1)^{\backslash \text{invert}\{I\}} \backslash \text{per } \mathbf{A}^{\backslash \text{lambda}}(I|I) \backslash \det \mathbf{A}^{\backslash \text{lambda}} \\ \backslash \det \mathbf{A}^{\backslash \text{lambda}} \\ (\overline{I} \cup \{l\} | \overline{I} \cup \{l\}). \\ \backslash \text{label}\{\text{sum-ali}\} \\ \backslash \text{end}\{\text{multline}\} \end{array}$

$\frac{37}{38}$  By (3), (6), and (7), we have

$\frac{39}{40}$  PROPOSITION 3.5.

$$(17) \quad H_c = \frac{1}{2n} \sum_{l=0}^n (-1)^l D_l,$$

$\frac{42}{43}$

1 where

$$2$$

$$3 \quad (18) \quad D_l = \sum_{I_l \subseteq \mathbf{n}} D(t_1, \dots, t_n) 2^{|t_i|} \Big|_{t_i = \begin{cases} 0, & \text{if } i \in I_l \\ 1, & \text{otherwise} \end{cases}, i=1, \dots, n}.$$

$$4$$

5

6

#### 7 4. Application

8 We consider here the applications of Theorems 5.1 and 5.2 to a complete  
9 multipartite graph  $K_{n_1 \dots n_p}$ . It can be shown that the number of spanning trees  
10 of  $K_{n_1 \dots n_p}$  may be written

$$11$$

$$12 \quad (19) \quad T = n^{p-2} \prod_{i=1}^p (n - n_i)^{n_i - 1}$$

$$13$$

14 where

$$15$$

$$16 \quad (20) \quad n = n_1 + \dots + n_p.$$

$$17$$

18 It follows from Theorems 5.1 and 5.2 that

$$19$$

$$20 \quad H_c = \frac{1}{2n} \sum_{l=0}^n (-1)^l (n-l)^{p-2} \sum_{l_1 + \dots + l_p = l} \prod_{i=1}^p \binom{n_i}{l_i}$$

$$21 \quad (21) \quad \cdot [(n-l) - (n_i - l_i)]^{n_i - l_i} \cdot \left[ (n-l)^2 - \sum_{j=1}^p (n_i - l_i)^2 \right].$$

$$22$$

$$23$$

$$24$$

25 ... \binom{n\_i}{l\_i}

26 and

$$27$$

$$28 \quad H_c = \frac{1}{2} \sum_{l=0}^{n-1} (-1)^l (n-l)^{p-2} \sum_{l_1 + \dots + l_p = l} \prod_{i=1}^p \binom{n_i}{l_i}$$

$$29 \quad (22) \quad \cdot [(n-l) - (n_i - l_i)]^{n_i - l_i} \left( 1 - \frac{l_p}{n_p} \right) [(n-l) - (n_p - l_p)].$$

$$30$$

$$31$$

$$32$$

33 The enumeration of  $H_c$  in a  $K_{n_1 \dots n_p}$  graph can also be carried out by  
34 Theorem 7.2 or 7.3 together with the algebraic method of (2). Some elegant  
35 representations may be obtained. For example,  $H_c$  in a  $K_{n_1 n_2 n_3}$  graph may be  
36 written

$$37$$

$$38 \quad H_c = \frac{n_1! n_2! n_3!}{n_1 + n_2 + n_3} \sum_i \left[ \binom{n_1}{i} \binom{n_2}{n_3 - n_1 + i} \binom{n_3}{n_3 - n_2 + i} \right.$$

$$39 \quad (23) \quad \left. + \binom{n_1 - 1}{i} \binom{n_2 - 1}{n_3 - n_1 + i} \binom{n_3 - 1}{n_3 - n_2 + i} \right].$$

$$40$$

$$41$$

$$42$$

## 5. Secret key exchanges

Modern cryptography is fundamentally concerned with the problem of secure private communication. A Secret Key Exchange is a protocol where Alice and Bob, having no secret information in common to start, are able to agree on a common secret key, conversing over a public channel. The notion of a Secret Key Exchange protocol was first introduced in the seminal paper of Diffie and Hellman [DH76]. [DH76] presented a concrete implementation of a Secret Key Exchange protocol, dependent on a specific assumption (a variant on the discrete log), specially tailored to yield Secret Key Exchange. Secret Key Exchange is of course trivial if trapdoor permutations exist. However, there is no known implementation based on a weaker general assumption.

The concept of an informationally one-way function was introduced in [ILL89]. We give only an informal definition here:

*Definition 5.1* (one way). A polynomial time computable function  $f = \{f_k\}$  is informationally one-way if there is no probabilistic polynomial time algorithm which (with probability of the form  $1 - k^{-e}$  for some  $e > 0$ ) returns on input  $y \in \{0, 1\}^k$  a random element of  $f^{-1}(y)$ .

In the non-uniform setting [ILL89] show that these are not weaker than one-way functions:

**THEOREM 5.1** ([ILL89] (non-uniform)). *The existence of informationally one-way functions implies the existence of one-way functions.*

We will stick to the convention introduced above of saying “non-uniform” before the theorem statement when the theorem makes use of non-uniformity. It should be understood that if nothing is said then the result holds for both the uniform and the non-uniform models.

It now follows from Theorem 5.1 that

**THEOREM 5.2** (non-uniform). *Weak SKE implies the existence of a one-way function.*

More recently, the polynomial-time, interior point algorithms for linear programming have been extended to the case of convex quadratic programs [MA87, Ye87], certain linear complementarity problems [KMY87b, MYK88], and the nonlinear complementarity problem [KMY87a]. The connection between these algorithms and the classical Newton method for nonlinear equations is well explained in [KMY87b].

## 6. Review

We begin our discussion with the following definition:

1        *Definition 6.1.* A function  $H: \mathfrak{R}^n \rightarrow \mathfrak{R}^n$  is said to be *B-differentiable*  
2 at the point  $z$  if (i)  $H$  is Lipschitz continuous in a neighborhood of  $z$ , and  
3 (ii) there exists a positive homogeneous function  $BH(z): \mathfrak{R}^n \rightarrow \mathfrak{R}^n$ , called the  
4 *B-derivative* of  $H$  at  $z$ , such that

$$\lim_{v \rightarrow 0} \frac{H(z+v) - H(z) - BH(z)v}{\|v\|} = 0.$$

7 The function  $H$  is *B-differentiable in set  $S$*  if it is B-differentiable at every  
8 point in  $S$ . The B-derivative  $BH(z)$  is said to be *strong* if

$$\lim_{(v,v') \rightarrow (0,0)} \frac{H(z+v) - H(z+v') - BH(z)(v-v')}{\|v-v'\|} = 0.$$

12        LEMMA 6.1. *There exists a smooth function  $\psi_0(z)$  defined for  $|z| > 1 - 2a$*   
13 *satisfying the following properties:*

- 14        (i)  $\psi_0(z)$  is bounded above and below by positive constants  $c_1 \leq \psi_0(z) \leq c_2$ .
- 15        (ii) If  $|z| > 1$ , then  $\psi_0(z) = 1$ .
- 16        (iii) For all  $z$  in the domain of  $\psi_0$ ,  $\Delta_0 \ln \psi_0 \geq 0$ .
- 17        (iv) If  $1 - 2a < |z| < 1 - a$ , then  $\Delta_0 \ln \psi_0 \geq c_3 > 0$ .

19        *Proof.* We choose  $\psi_0(z)$  to be a radial function depending only on  $r = |z|$ .  
20 Let  $h(r) \geq 0$  be a suitable smooth function satisfying  $h(r) \geq c_3$  for  $1 - 2a <$   
21  $|z| < 1 - a$ , and  $h(r) = 0$  for  $|z| > 1 - \frac{a}{2}$ . The radial Laplacian

$$\Delta_0 \ln \psi_0(r) = \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \ln \psi_0(r)$$

24 has smooth coefficients for  $r > 1 - 2a$ . Therefore, we may apply the existence  
25 and uniqueness theory for ordinary differential equations. Simply let  $\ln \psi_0(r)$   
26 be the solution of the differential equation

$$\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \ln \psi_0(r) = h(r)$$

28 with initial conditions given by  $\ln \psi_0(1) = 0$  and  $\ln \psi_0'(1) = 0$ .

31        Next, let  $D_\nu$  be a finite collection of pairwise disjoint disks, all of which  
32 are contained in the unit disk centered at the origin in  $C$ . We assume that  
33  $D_\nu = \{z \mid |z - z_\nu| < \delta\}$ . Suppose that  $D_\nu(a)$  denotes the smaller concentric  
34 disk  $D_\nu(a) = \{z \mid |z - z_\nu| \leq (1 - 2a)\delta\}$ . We define a smooth weight function  
35  $\Phi_0(z)$  for  $z \in C - \bigcup_\nu D_\nu(a)$  by setting  $\Phi_0(z) = 1$  when  $z \notin \bigcup_\nu D_\nu$  and  $\Phi_0(z) =$   
36  $\psi_0((z - z_\nu)/\delta)$  when  $z$  is an element of  $D_\nu$ . It follows from Lemma 6.1 that  $\Phi_0$   
37 satisfies the properties:

- 38        (i)  $\Phi_0(z)$  is bounded above and below by positive constants  $c_1 \leq \Phi_0(z) \leq$   
39  $c_2$ .
- 40        (ii)  $\Delta_0 \ln \Phi_0 \geq 0$  for all  $z \in C - \bigcup_\nu D_\nu(a)$ , the domain where the function  
41  $\Phi_0$  is defined.



$\frac{1}{2}$  (iii)  $\Delta_0 \ln \Phi_0 \geq c_3 \delta^{-2}$  when  $(1 - 2a)\delta < |z - z_\nu| < (1 - a)\delta$ .

$\frac{2}{3}$  Let  $A_\nu$  denote the annulus  $A_\nu = \{(1 - 2a)\delta < |z - z_\nu| < (1 - a)\delta\}$ , and  
 $\frac{3}{4}$  set  $A = \bigcup_\nu A_\nu$ . The properties (2) and (3) of  $\Phi_0$  may be summarized as  
 $\frac{4}{5}$   $\Delta_0 \ln \Phi_0 \geq c_3 \delta^{-2} \chi_A$ , where  $\chi_A$  is the characteristic function of  $A$ .  $\square$

$\frac{6}{7}$  Suppose that  $\alpha$  is a nonnegative real constant. We apply Proposition 3.5  
 $\frac{7}{8}$  with  $\Phi(z) = \Phi_0(z)e^{\alpha|z|^2}$ . If  $u \in C_0^\infty(R^2 - \bigcup_\nu D_\nu(a))$ , assume that  $\mathcal{D}$  is a  
 $\frac{8}{9}$  bounded domain containing the support of  $u$  and  $A \subset \mathcal{D} \subset R^2 - \bigcup_\nu D_\nu(a)$ . A  
 $\frac{9}{10}$  calculation gives

$$\frac{10}{11} \int_{\mathcal{D}} |\bar{\partial}u|^2 \Phi_0(z)e^{\alpha|z|^2} \geq c_4 \alpha \int_{\mathcal{D}} |u|^2 \Phi_0 e^{\alpha|z|^2} + c_5 \delta^{-2} \int_A |u|^2 \Phi_0 e^{\alpha|z|^2}.$$

$\frac{12}{13}$  The boundedness, property (1) of  $\Phi_0$ , then yields

$$\frac{14}{15} \int_{\mathcal{D}} |\bar{\partial}u|^2 e^{\alpha|z|^2} \geq c_6 \alpha \int_{\mathcal{D}} |u|^2 e^{\alpha|z|^2} + c_7 \delta^{-2} \int_A |u|^2 e^{\alpha|z|^2}.$$

$\frac{16}{17}$  Let  $B(X)$  be the set of blocks of  $\Lambda_X$  and let  $b(X) = |B(X)|$ . If  $\phi \in Q_X$   
 $\frac{17}{18}$  then  $\phi$  is constant on the blocks of  $\Lambda_X$ .

$$\frac{19}{20} (24) \quad P_X = \{\phi \in M \mid \Lambda_\phi = \Lambda_X\}, \quad Q_X = \{\phi \in M \mid \Lambda_\phi \geq \Lambda_X\}.$$

$\frac{20}{21}$  If  $\Lambda_\phi \geq \Lambda_X$  then  $\Lambda_\phi = \Lambda_Y$  for some  $Y \geq X$  so that

$$\frac{22}{23} Q_X = \bigcup_{Y \geq X} P_Y.$$

$\frac{24}{25}$  Thus by Möbius inversion

$$\frac{26}{27} |P_Y| = \sum_{X \geq Y} \mu(Y, X) |Q_X|.$$

$\frac{28}{29}$  Thus there is a bijection from  $Q_X$  to  $W^{B(X)}$ . In particular  $|Q_X| = w^{b(X)}$ .

$\frac{30}{31}$  Next note that  $b(X) = \dim X$ . We see this by choosing a basis for  $X$   
 $\frac{31}{32}$  consisting of vectors  $v^k$  defined by

$$\frac{32}{33} v_i^k = \begin{cases} 1 & \text{if } i \in \Lambda_k, \\ 0 & \text{otherwise.} \end{cases}$$

$\frac{34}{35}$   $\backslash [v^{\{k\}}_i =$

$\frac{35}{36}$   $\backslash \begin{cases} 1 & \text{if } i \in \Lambda_{\{k\}}, \\ 0 & \text{otherwise.} \end{cases} \backslash \backslash$

$\frac{36}{37}$   $\backslash \end{cases} \backslash \backslash$

$\frac{37}{38}$   $\backslash ]$

$\frac{39}{40}$  LEMMA 6.2. *Let  $\mathcal{A}$  be an arrangement. Then*

$$\frac{40}{41} \chi(\mathcal{A}, t) = \sum_{\mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|} t^{\dim T(\mathcal{B})}.$$

$\frac{42}{43}$

1 In order to compute  $R''$  recall the definition of  $S(X, Y)$  from Lemma 3.1.  
2 Since  $H \in \mathcal{B}$ ,  $\mathcal{A}_H \subseteq \mathcal{B}$ . Thus if  $T(\mathcal{B}) = Y$  then  $\mathcal{B} \in S(H, Y)$ . Let  $L'' = L(\mathcal{A}'')$ .  
3 Then

$$\begin{aligned}
 \frac{4}{R''} &= \sum_{H \in \mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|} t^{\dim T(\mathcal{B})} \\
 \frac{5}{} &= \sum_{Y \in L''} \sum_{\mathcal{B} \in S(H, Y)} (-1)^{|\mathcal{B}|} t^{\dim Y} \\
 \frac{6}{} &= - \sum_{Y \in L''} \sum_{\mathcal{B} \in S(H, Y)} (-1)^{|\mathcal{B} - \mathcal{A}_H|} t^{\dim Y} \\
 \frac{7}{(25)} &= - \sum_{Y \in L''} \mu(H, Y) t^{\dim Y} \\
 \frac{8}{} &= -\chi(\mathcal{A}'', t).
 \end{aligned}$$

14 COROLLARY 6.3. Let  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  be a triple of arrangements. Then

$$\pi(\mathcal{A}, t) = \pi(\mathcal{A}', t) + t\pi(\mathcal{A}'', t).$$

17 Definition 6.2. Let  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  be a triple with respect to the hyperplane  
18  $H \in \mathcal{A}$ . Call  $H$  a separator if  $T(\mathcal{A}) \notin L(\mathcal{A}')$ .

20 COROLLARY 6.4. Let  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  be a triple with respect to  $H \in \mathcal{A}$ .

21 (i) If  $H$  is a separator then

$$\mu(\mathcal{A}) = -\mu(\mathcal{A}'')$$

24 and hence

$$|\mu(\mathcal{A})| = |\mu(\mathcal{A}'')|.$$

26 (ii) If  $H$  is not a separator then

$$\mu(\mathcal{A}) = \mu(\mathcal{A}') - \mu(\mathcal{A}'')$$

29 and

$$|\mu(\mathcal{A})| = |\mu(\mathcal{A}')| + |\mu(\mathcal{A}'')|.$$

32 Proof. It follows from Theorem 5.1 that  $\pi(\mathcal{A}, t)$  has leading term

$$(-1)^{r(\mathcal{A})} \mu(\mathcal{A}) t^{r(\mathcal{A})}.$$

35 The conclusion follows by comparing coefficients of the leading terms on both  
36 sides of the equation in Corollary 6.3. If  $H$  is a separator then  $r(\mathcal{A}') < r(\mathcal{A})$   
37 and there is no contribution from  $\pi(\mathcal{A}', t)$ .  $\square$

38 The Poincaré polynomial of an arrangement will appear repeatedly in  
39 these notes. It will be shown to equal the Poincaré polynomial of the graded  
40 algebras which we are going to associate with  $\mathcal{A}$ . It is also the Poincaré poly-  
41 nomial of the complement  $M(\mathcal{A})$  for a complex arrangement. Here we prove  
42

1  
2  
3  
4  
5  
6  
7  
8  
9  
10  
11  
12  
13  
14  
15  
16  
17  
18  
19  
20  
21  
22  
23  
24  
25  
26  
27  
28  
29  
30  
31  
32  
33  
34  
35  
36  
37  
38  
39  
40  
41  
42

Figure 1.  $Q(\mathcal{A}_1) = xyz(x - z)(x + z)(y - z)(y + z)$

Figure 2.  $Q(\mathcal{A}_2) = xyz(x + y + z)(x + y - z)(x - y + z)(x - y - z)$

that the Poincaré polynomial is the chamber counting function for a real arrangement. The complement  $M(\mathcal{A})$  is a disjoint union of chambers

$$M(\mathcal{A}) = \bigcup_{C \in \text{Cham}(\mathcal{A})} C.$$

The number of chambers is determined by the Poincaré polynomial as follows.

**THEOREM 6.5.** *Let  $\mathcal{A}_{\mathbf{R}}$  be a real arrangement. Then*

$$|\text{Cham}(\mathcal{A}_{\mathbf{R}})| = \pi(\mathcal{A}_{\mathbf{R}}, 1).$$

*Proof.* We check the properties required in Corollary 6.4: (i) follows from  $\pi(\Phi_l, t) = 1$ , and (ii) is a consequence of Corollary 3.4.  $\square$

**THEOREM 6.6.** *Let  $\phi$  be a protocol for a random pair  $(X, Y)$ . If one of  $\sigma_\phi(x', y)$  and  $\sigma_\phi(x, y')$  is a prefix of the other and  $(x, y) \in S_{X, Y}$ , then*

$$\langle \sigma_j(x', y) \rangle_{j=1}^\infty = \langle \sigma_j(x, y) \rangle_{j=1}^\infty = \langle \sigma_j(x, y') \rangle_{j=1}^\infty.$$

1 *Proof.* We show by induction on  $i$  that

$$\langle \sigma_j(x', y) \rangle_{j=1}^i = \langle \sigma_j(x, y) \rangle_{j=1}^i = \langle \sigma_j(x, y') \rangle_{j=1}^i.$$

4 The induction hypothesis holds vacuously for  $i = 0$ . Assume it holds for  
5  $i - 1$ , in particular  $[\sigma_j(x', y)]_{j=1}^{i-1} = [\sigma_j(x, y')]_{j=1}^{i-1}$ . Then one of  $[\sigma_j(x', y)]_{j=i}^\infty$   
6 and  $[\sigma_j(x, y')]_{j=i}^\infty$  is a prefix of the other which implies that one of  $\sigma_i(x', y)$   
7 and  $\sigma_i(x, y')$  is a prefix of the other. If the  $i$ th message is transmitted by  
8  $P_X$  then, by the separate-transmissions property and the induction hypothe-  
9 sis,  $\sigma_i(x, y) = \sigma_i(x, y')$ , hence one of  $\sigma_i(x, y)$  and  $\sigma_i(x', y)$  is a prefix of the  
10 other. By the implicit-termination property, neither  $\sigma_i(x, y)$  nor  $\sigma_i(x', y)$  can  
11 be a proper prefix of the other, hence they must be the same and  $\sigma_i(x', y) =$   
12  $\sigma_i(x, y) = \sigma_i(x, y')$ . If the  $i$ th message is transmitted by  $P_Y$  then, symmet-  
13 rically,  $\sigma_i(x, y) = \sigma_i(x', y)$  by the induction hypothesis and the separate-  
14 transmissions property, and, then,  $\sigma_i(x, y) = \sigma_i(x, y')$  by the implicit-termination  
15 property, proving the induction step.  $\square$

16 If  $\phi$  is a protocol for  $(X, Y)$ , and  $(x, y), (x', y)$  are distinct inputs in  $S_{X,Y}$ ,  
17 then, by the correct-decision property,  $\langle \sigma_j(x, y) \rangle_{j=1}^\infty \neq \langle \sigma_j(x', y) \rangle_{j=1}^\infty$ .

18 Equation (25) defined  $P_Y$ 's ambiguity set  $S_{X|Y}(y)$  to be the set of possible  
19  $X$  values when  $Y = y$ . The last corollary implies that for all  $y \in S_Y$ , the  
20 multiset<sup>1</sup> of codewords  $\{\sigma_\phi(x, y) : x \in S_{X|Y}(y)\}$  is prefix free.

21

22

## 7. One-way complexity

23

24  $\hat{C}_1(X|Y)$ , the one-way complexity of a random pair  $(X, Y)$ , is the number  
25 of bits  $P_X$  must transmit in the worst case when  $P_Y$  is not permitted to transmit  
26 any feedback messages. Starting with  $S_{X,Y}$ , the support set of  $(X, Y)$ , we define  
27  $G(X|Y)$ , the *characteristic hypergraph* of  $(X, Y)$ , and show that

28

29

$$\hat{C}_1(X|Y) = \lceil \log \chi(G(X|Y)) \rceil .$$

30 Let  $(X, Y)$  be a random pair. For each  $y$  in  $S_Y$ , the support set of  $Y$ ,  
31 equation (25) defined  $S_{X|Y}(y)$  to be the set of possible  $x$  values when  $Y = y$ .  
32 The *characteristic hypergraph*  $G(X|Y)$  of  $(X, Y)$  has  $S_X$  as its vertex set and  
33 the hyperedge  $S_{X|Y}(y)$  for each  $y \in S_Y$ .

34 We can now prove a continuity theorem.

35

36

**THEOREM 7.1.** *Let  $\Omega \subset \mathbf{R}^n$  be an open set, let  $u \in BV(\Omega; \mathbf{R}^m)$ , and let*

37

38

$$(26) \quad T_x^u = \left\{ y \in \mathbf{R}^m : y = \tilde{u}(x) + \left\langle \frac{Du}{|Du|}(x), z \right\rangle \text{ for some } z \in \mathbf{R}^n \right\}$$

39

40

41

42

<sup>1</sup>A multiset allows multiplicity of elements. Hence,  $\{0, 01, 01\}$  is prefix free as a set, but not as a multiset.

$\frac{1}{2}$  for every  $x \in \Omega \setminus S_u$ . Let  $f: \mathbf{R}^m \rightarrow \mathbf{R}^k$  be a Lipschitz continuous function such  
 $\frac{2}{2}$  that  $f(0) = 0$ , and let  $v = f(u): \Omega \rightarrow \mathbf{R}^k$ . Then  $v \in BV(\Omega; \mathbf{R}^k)$  and

$\frac{3}{4}$  (27) 
$$Jv = (f(u^+) - f(u^-)) \otimes \nu_u \cdot \mathcal{H}_{n-1} \Big|_{S_u}.$$

$\frac{5}{6}$  In addition, for  $|\tilde{D}u|$ -almost every  $x \in \Omega$  the restriction of the function  $f$  to  
 $\frac{7}{7}$   $T_x^u$  is differentiable at  $\tilde{u}(x)$  and

$\frac{8}{9}$  (28) 
$$\tilde{D}v = \nabla(f|_{T_x^u})(\tilde{u}) \frac{\tilde{D}u}{|\tilde{D}u|} \cdot |\tilde{D}u|.$$

$\frac{11}{12}$  Before proving the theorem, we state without proof three elementary remarks which will be useful in the sequel.

$\frac{13}{14}$  *Remark 7.1.* Let  $\omega: ]0, +\infty[ \rightarrow ]0, +\infty[$  be a continuous function such  
 $\frac{15}{15}$  that  $\omega(t) \rightarrow 0$  as  $t \rightarrow 0$ . Then

$\frac{16}{17}$  
$$\lim_{h \rightarrow 0^+} g(\omega(h)) = L \Leftrightarrow \lim_{h \rightarrow 0^+} g(h) = L$$

$\frac{18}{19}$  for any function  $g: ]0, +\infty[ \rightarrow \mathbf{R}$ .

$\frac{20}{21}$  *Remark 7.2.* Let  $g: \mathbf{R}^n \rightarrow \mathbf{R}$  be a Lipschitz continuous function and assume that

$\frac{22}{23}$  
$$L(z) = \lim_{h \rightarrow 0^+} \frac{g(hz) - g(0)}{h}$$

$\frac{24}{25}$  exists for every  $z \in \mathbf{Q}^n$  and that  $L$  is a linear function of  $z$ . Then  $g$  is differentiable at 0.

$\frac{26}{27}$  *Remark 7.3.* Let  $A: \mathbf{R}^n \rightarrow \mathbf{R}^m$  be a linear function, and let  $f: \mathbf{R}^m \rightarrow \mathbf{R}$   
 $\frac{28}{29}$  be a function. Then the restriction of  $f$  to the range of  $A$  is differentiable at 0 if and only if  $f(A): \mathbf{R}^n \rightarrow \mathbf{R}$  is differentiable at 0 and

$\frac{30}{31}$  
$$\nabla(f|_{\text{Im}(A)})(0)A = \nabla(f(A))(0).$$

$\frac{32}{32}$  *Proof.* We begin by showing that  $v \in BV(\Omega; \mathbf{R}^k)$  and

$\frac{33}{34}$  (29) 
$$|Dv|(B) \leq K |Du|(B) \quad \forall B \in \mathbf{B}(\Omega),$$

$\frac{35}{36}$  where  $K > 0$  is the Lipschitz constant of  $f$ . By (13) and by the approximation result quoted in §3, it is possible to find a sequence  $(u_h) \subset C^1(\Omega; \mathbf{R}^m)$   
 $\frac{37}{37}$  converging to  $u$  in  $L^1(\Omega; \mathbf{R}^m)$  and such that

$\frac{38}{39}$  
$$\lim_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h| \, dx = |Du|(\Omega).$$

$\frac{40}{41}$  The functions  $v_h = f(u_h)$  are locally Lipschitz continuous in  $\Omega$ , and the definition of differential implies that  $|\nabla v_h| \leq K |\nabla u_h|$  almost everywhere in  $\Omega$ . The  
 $\frac{42}{42}$

1 lower semicontinuity of the total variation and (13) yield

$$\begin{aligned}
2 \\
3 \quad |Dv|(\Omega) &\leq \liminf_{h \rightarrow +\infty} |Dv_h|(\Omega) = \liminf_{h \rightarrow +\infty} \int_{\Omega} |\nabla v_h| \, dx \\
4 \quad (30) \\
5 &\leq K \liminf_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h| \, dx = K |Du|(\Omega). \\
6
\end{aligned}$$

7 Since  $f(0) = 0$ , we have also

$$\begin{aligned}
8 \\
9 \quad \int_{\Omega} |v| \, dx &\leq K \int_{\Omega} |u| \, dx; \\
10
\end{aligned}$$

11 therefore  $u \in BV(\Omega; \mathbf{R}^k)$ . Repeating the same argument for every open set  
12  $A \subset \Omega$ , we get (29) for every  $B \in \mathbf{B}(\Omega)$ , because  $|Dv|$ ,  $|Du|$  are Radon mea-  
13 sures. To prove Lemma 6.1, first we observe that

$$\begin{aligned}
14 \quad (31) \quad S_v \subset S_u, \quad \tilde{v}(x) = f(\tilde{u}(x)) \quad \forall x \in \Omega \setminus S_u. \\
15
\end{aligned}$$

16 In fact, for every  $\varepsilon > 0$  we have

$$\begin{aligned}
17 \\
18 \quad \{y \in B_{\rho}(x) : |v(y) - f(\tilde{u}(x))| > \varepsilon\} &\subset \{y \in B_{\rho}(x) : |u(y) - \tilde{u}(x)| > \varepsilon/K\}, \\
19
\end{aligned}$$

19 hence

$$\begin{aligned}
20 \\
21 \quad \lim_{\rho \rightarrow 0^+} \frac{|\{y \in B_{\rho}(x) : |v(y) - f(\tilde{u}(x))| > \varepsilon\}|}{\rho^n} = 0 \\
22
\end{aligned}$$

23 whenever  $x \in \Omega \setminus S_u$ . By a similar argument, if  $x \in S_u$  is a point such that  
24 there exists a triplet  $(u^+, u^-, \nu_u)$  satisfying (14), (15), then

$$\begin{aligned}
25 \\
26 \quad (v^+(x) - v^-(x)) \otimes \nu_v &= (f(u^+(x)) - f(u^-(x))) \otimes \nu_u \quad \text{if } x \in S_v
\end{aligned}$$

27 and  $f(u^-(x)) = f(u^+(x))$  if  $x \in S_u \setminus S_v$ . Hence, by (1.8) we get

$$\begin{aligned}
28 \\
29 \quad Jv(B) &= \int_{B \cap S_v} (v^+ - v^-) \otimes \nu_v \, d\mathcal{H}_{n-1} = \int_{B \cap S_v} (f(u^+) - f(u^-)) \otimes \nu_u \, d\mathcal{H}_{n-1} \\
30 \\
31 &= \int_{B \cap S_u} (f(u^+) - f(u^-)) \otimes \nu_u \, d\mathcal{H}_{n-1} \\
32
\end{aligned}$$

33 and Lemma 6.1 is proved. □

34

35 To prove (31), it is not restrictive to assume that  $k = 1$ . Moreover, to  
36 simplify our notation, from now on we shall assume that  $\Omega = \mathbf{R}^n$ . The proof  
37 of (31) is divided into two steps. In the first step we prove the statement in  
38 the one-dimensional case ( $n = 1$ ), using Theorem 5.2. In the second step we  
39 achieve the general result using Theorem 7.1.

40 *Step 1.* Assume that  $n = 1$ . Since  $S_u$  is at most countable, (7) yields  
41 that  $|\tilde{D}v|(S_u \setminus S_v) = 0$ , so that (19) and (21) imply that  $Dv = \tilde{D}v + Jv$  is the  
42

$\frac{1}{2}$  Radon-Nikodým decomposition of  $Dv$  in absolutely continuous and singular  
 $\frac{2}{3}$  part with respect to  $|\widetilde{D}u|$ . By Theorem 5.2, we have

$$\frac{4}{5} \quad \frac{\widetilde{D}v}{|\widetilde{D}u|}(t) = \lim_{s \rightarrow t^+} \frac{Dv([t, s[)}{|\widetilde{D}u|([t, s[)}, \quad \frac{\widetilde{D}u}{|\widetilde{D}u|}(t) = \lim_{s \rightarrow t^+} \frac{Du([t, s[)}{|\widetilde{D}u|([t, s[)}$$

$\frac{7}{8}$   $|\widetilde{D}u|$ -almost everywhere in  $\mathbf{R}$ . It is well known (see, for instance, [Ste70,  
 $\frac{8}{9}$  2.5.16]) that every one-dimensional function of bounded variation  $w$  has a  
 $\frac{9}{10}$  unique left continuous representative, i.e., a function  $\hat{w}$  such that  $\hat{w} = w$   
 $\frac{10}{11}$  almost everywhere and  $\lim_{s \rightarrow t^-} \hat{w}(s) = \hat{w}(t)$  for every  $t \in \mathbf{R}$ . These conditions  
 $\frac{11}{12}$  imply

$$\frac{13}{14} \quad (32) \quad \hat{u}(t) = Du(]-\infty, t]), \quad \hat{v}(t) = Dv(]-\infty, t]) \quad \forall t \in \mathbf{R}$$

$\frac{15}{16}$  and

$$\frac{17}{18} \quad (33) \quad \hat{v}(t) = f(\hat{u}(t)) \quad \forall t \in \mathbf{R}.$$

$\frac{18}{19}$  Let  $t \in \mathbf{R}$  be such that  $|\widetilde{D}u|([t, s]) > 0$  for every  $s > t$  and assume that the  
 $\frac{19}{20}$  limits in (22) exist. By (23) and (24) we get

$$\begin{aligned} \frac{21}{22} \quad \frac{\hat{v}(s) - \hat{v}(t)}{|\widetilde{D}u|([t, s])} &= \frac{f(\hat{u}(s)) - f(\hat{u}(t))}{|\widetilde{D}u|([t, s])} \\ \frac{23}{24} &= \frac{f(\hat{u}(s)) - f(\hat{u}(t) + \frac{\widetilde{D}u}{|\widetilde{D}u|}(t) |\widetilde{D}u|([t, s]))}{|\widetilde{D}u|([t, s])} \\ \frac{25}{26} &= \frac{f(\hat{u}(s)) - f(\hat{u}(t) + \frac{\widetilde{D}u}{|\widetilde{D}u|}(t) |\widetilde{D}u|([t, s])) - f(\hat{u}(t))}{|\widetilde{D}u|([t, s])} \\ \frac{27}{28} &+ \frac{f(\hat{u}(t) + \frac{\widetilde{D}u}{|\widetilde{D}u|}(t) |\widetilde{D}u|([t, s])) - f(\hat{u}(t))}{|\widetilde{D}u|([t, s])} \end{aligned}$$

$\frac{31}{32}$  for every  $s > t$ . Using the Lipschitz condition on  $f$  we find

$$\begin{aligned} \frac{34}{35} \quad &\left| \frac{\hat{v}(s) - \hat{v}(t)}{|\widetilde{D}u|([t, s])} - \frac{f(\hat{u}(t) + \frac{\widetilde{D}u}{|\widetilde{D}u|}(t) |\widetilde{D}u|([t, s])) - f(\hat{u}(t))}{|\widetilde{D}u|([t, s])} \right| \\ \frac{36}{37} & \\ \frac{38}{39} & \\ \frac{40}{41} & \leq K \left| \frac{\hat{u}(s) - \hat{u}(t)}{|\widetilde{D}u|([t, s])} - \frac{\widetilde{D}u}{|\widetilde{D}u|}(t) \right|. \\ \frac{42}{43} & \end{aligned}$$

$\frac{1}{2}$  By (29), the function  $s \rightarrow |\widetilde{D}u|([t, s])$  is continuous and converges to 0 as  $s \downarrow t$ .  
 $\frac{2}{2}$  Therefore Remark 7.1 and the previous inequality imply

$$\frac{3}{4} \quad \frac{4}{5} \quad \frac{5}{6} \quad \frac{6}{7} \quad \frac{\widetilde{D}v}{|\widetilde{D}u|}(t) = \lim_{h \rightarrow 0^+} \frac{f(\hat{u}(t) + h \frac{\widetilde{D}u}{|\widetilde{D}u|}(t)) - f(\hat{u}(t))}{h} \quad |\widetilde{D}u| \text{-a.e. in } \mathbf{R}.$$

$\frac{8}{8}$  By (22),  $\hat{u}(x) = \tilde{u}(x)$  for every  $x \in \mathbf{R} \setminus S_u$ ; moreover, applying the same argu-  
 $\frac{9}{9}$  ment to the functions  $u'(t) = u(-t)$ ,  $v'(t) = f(u'(t)) = v(-t)$ , we get

$$\frac{10}{11} \quad \frac{11}{12} \quad \frac{12}{13} \quad \frac{13}{14} \quad \frac{\widetilde{D}v}{|\widetilde{D}u|}(t) = \lim_{h \rightarrow 0} \frac{f(\tilde{u}(t) + h \frac{\widetilde{D}u}{|\widetilde{D}u|}(t)) - f(\tilde{u}(t))}{h} \quad |\widetilde{D}u| \text{-a.e. in } \mathbf{R}$$

$\frac{15}{15}$  and our statement is proved.

$\frac{16}{16}$  *Step 2.* Let us consider now the general case  $n > 1$ . Let  $\nu \in \mathbf{R}^n$  be such  
 $\frac{17}{17}$  that  $|\nu| = 1$ , and let  $\pi_\nu = \{y \in \mathbf{R}^n : \langle y, \nu \rangle = 0\}$ . In the following, we shall  
 $\frac{18}{18}$  identify  $\mathbf{R}^n$  with  $\pi_\nu \times \mathbf{R}$ , and we shall denote by  $y$  the variable ranging in  $\pi_\nu$   
 $\frac{19}{19}$  and by  $t$  the variable ranging in  $\mathbf{R}$ . By the just proven one-dimensional result,  
 $\frac{20}{20}$  and by Theorem 3.3, we get

$$\frac{21}{22} \quad \frac{22}{23} \quad \frac{23}{24} \quad \frac{24}{25} \quad \lim_{h \rightarrow 0} \frac{f(\tilde{u}(y + t\nu) + h \frac{\widetilde{D}u_y}{|\widetilde{D}u_y|}(t)) - f(\tilde{u}(y + t\nu))}{h} = \frac{\widetilde{D}v_y}{|\widetilde{D}u_y|}(t) \quad |\widetilde{D}u_y| \text{-a.e. in } \mathbf{R}$$

$\frac{26}{26}$  for  $\mathcal{H}_{n-1}$ -almost every  $y \in \pi_\nu$ . We claim that

$$\frac{27}{28} \quad \frac{28}{29} \quad \frac{29}{30} \quad (34) \quad \frac{\langle \widetilde{D}u, \nu \rangle}{|\langle \widetilde{D}u, \nu \rangle|}(y + t\nu) = \frac{\widetilde{D}u_y}{|\widetilde{D}u_y|}(t) \quad |\widetilde{D}u_y| \text{-a.e. in } \mathbf{R}$$

$\frac{31}{31}$  for  $\mathcal{H}_{n-1}$ -almost every  $y \in \pi_\nu$ . In fact, by (16) and (18) we get

$$\frac{32}{33} \quad \frac{33}{34} \quad \frac{34}{35} \quad \int_{\pi_\nu} \frac{\widetilde{D}u_y}{|\widetilde{D}u_y|} \cdot |\widetilde{D}u_y| d\mathcal{H}_{n-1}(y) = \int_{\pi_\nu} \widetilde{D}u_y d\mathcal{H}_{n-1}(y)$$

$$\frac{36}{37} \quad = \langle \widetilde{D}u, \nu \rangle = \frac{\langle \widetilde{D}u, \nu \rangle}{|\langle \widetilde{D}u, \nu \rangle|} \cdot |\langle \widetilde{D}u, \nu \rangle| = \int_{\pi_\nu} \frac{\langle \widetilde{D}u, \nu \rangle}{|\langle \widetilde{D}u, \nu \rangle|}(y + \cdot\nu) \cdot |\widetilde{D}u_y| d\mathcal{H}_{n-1}(y)$$

$\frac{38}{38}$  and (24) follows from (13). By the same argument it is possible to prove that

$$\frac{39}{40} \quad \frac{40}{41} \quad \frac{41}{42} \quad (35) \quad \frac{\langle \widetilde{D}v, \nu \rangle}{|\langle \widetilde{D}v, \nu \rangle|}(y + t\nu) = \frac{\widetilde{D}v_y}{|\widetilde{D}u_y|}(t) \quad |\widetilde{D}u_y| \text{-a.e. in } \mathbf{R}$$



$\frac{1}{2}$  for  $\mathcal{H}_{n-1}$ -almost every  $y \in \pi_\nu$ . By (24) and (25) we get

$$\frac{3}{4} \quad \lim_{h \rightarrow 0} \frac{f(\tilde{u}(y + t\nu) + h \frac{\langle \tilde{D}u, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|}(y + t\nu)) - f(\tilde{u}(y + t\nu))}{h} = \frac{\langle \tilde{D}v, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|}(y + t\nu)$$

$\frac{6}{7}$  for  $\mathcal{H}_{n-1}$ -almost every  $y \in \pi_\nu$ , and using again (14), (15) we get

$$\frac{8}{9} \quad \lim_{h \rightarrow 0} \frac{f(\tilde{u}(x) + h \frac{\langle \tilde{D}u, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|}(x)) - f(\tilde{u}(x))}{h} = \frac{\langle \tilde{D}v, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|}(x)$$

$\frac{12}{13}$   $|\langle \tilde{D}u, \nu \rangle|$ -a.e. in  $\mathbf{R}^n$ .

$\frac{14}{15}$  Since the function  $|\langle \tilde{D}u, \nu \rangle| / |\tilde{D}u|$  is strictly positive  $|\langle \tilde{D}u, \nu \rangle|$ -almost everywhere, we obtain also

$$\frac{16}{17} \quad \lim_{h \rightarrow 0} \frac{f(\tilde{u}(x) + h \frac{|\langle \tilde{D}u, \nu \rangle|}{|\tilde{D}u|}(x) \frac{\langle \tilde{D}u, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|}(x)) - f(\tilde{u}(x))}{h}$$

$$\frac{20}{21} \quad = \frac{|\langle \tilde{D}u, \nu \rangle|}{|\tilde{D}u|}(x) \frac{\langle \tilde{D}v, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|}(x)$$

$\frac{23}{24}$   $|\langle \tilde{D}u, \nu \rangle|$ -almost everywhere in  $\mathbf{R}^n$ .

$\frac{25}{26}$  Finally, since

$$\frac{26}{27} \quad \frac{|\langle \tilde{D}u, \nu \rangle|}{|\tilde{D}u|} \frac{\langle \tilde{D}u, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|} = \frac{\langle \tilde{D}u, \nu \rangle}{|\tilde{D}u|} = \left\langle \frac{\tilde{D}u}{|\tilde{D}u|}, \nu \right\rangle \quad |\tilde{D}u| \text{-a.e. in } \mathbf{R}^n$$

$$\frac{29}{30} \quad \frac{|\langle \tilde{D}u, \nu \rangle|}{|\tilde{D}u|} \frac{\langle \tilde{D}v, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|} = \frac{\langle \tilde{D}v, \nu \rangle}{|\tilde{D}u|} = \left\langle \frac{\tilde{D}v}{|\tilde{D}u|}, \nu \right\rangle \quad |\tilde{D}u| \text{-a.e. in } \mathbf{R}^n$$

$\frac{32}{33}$  and since both sides of (33) are zero  $|\tilde{D}u|$ -almost everywhere on  $|\langle \tilde{D}u, \nu \rangle|$ -negligible sets, we conclude that

$$\frac{35}{36} \quad \lim_{h \rightarrow 0} \frac{f\left(\tilde{u}(x) + h \left\langle \frac{\tilde{D}u}{|\tilde{D}u|}(x), \nu \right\rangle\right) - f(\tilde{u}(x))}{h} = \left\langle \frac{\tilde{D}v}{|\tilde{D}u|}(x), \nu \right\rangle,$$

$\frac{39}{40}$   $|\tilde{D}u|$ -a.e. in  $\mathbf{R}^n$ . Since  $\nu$  is arbitrary, by Remarks 7.2 and 7.3 the restriction of  $f$  to the affine space  $T_x^u$  is differentiable at  $\tilde{u}(x)$  for  $|\tilde{D}u|$ -almost every  $x \in \mathbf{R}^n$  and (26) holds.  $\square$

$\frac{42}{42}$

1 It follows from (13), (14), and (15) that

$$\begin{array}{l} \text{2} \\ \text{3} \end{array} \quad (36) \quad D(t_1, \dots, t_n) = \sum_{I \in \mathbf{n}} (-1)^{|I|-1} |I| \prod_{i \in I} t_i \prod_{j \in I} (D_j + \lambda_j t_j) \det \mathbf{A}^{(\lambda)}(\bar{I}|\bar{I}).$$

4  
5 Let  $t_i = \hat{x}_i$ ,  $i = 1, \dots, n$ . Lemma 1 leads to

$$\begin{array}{l} \text{6} \\ \text{7} \end{array} \quad (37) \quad D(\hat{x}_1, \dots, \hat{x}_n) = \prod_{i \in \mathbf{n}} \hat{x}_i \sum_{I \in \mathbf{n}} (-1)^{|I|-1} |I| \text{per } \mathbf{A}^{(\lambda)}(I|I) \det \mathbf{A}^{(\lambda)}(\bar{I}|\bar{I}).$$

8  
9 By (3), (13), and (37), we have the following result:

10 THEOREM 7.2.

$$\begin{array}{l} \text{11} \\ \text{12} \\ \text{13} \end{array} \quad (38) \quad H_c = \frac{1}{2n} \sum_{l=1}^n l (-1)^{l-1} A_l^{(\lambda)},$$

14 where

$$\begin{array}{l} \text{15} \\ \text{16} \\ \text{17} \end{array} \quad (39) \quad A_l^{(\lambda)} = \sum_{I_l \subseteq \mathbf{n}} \text{per } \mathbf{A}^{(\lambda)}(I_l|I_l) \det \mathbf{A}^{(\lambda)}(\bar{I}_l|\bar{I}_l), |I_l| = l.$$

18 It is worth noting that  $A_l^{(\lambda)}$  of (39) is similar to the coefficients  $b_l$  of the  
19 characteristic polynomial of (10). It is well known in graph theory that the  
20 coefficients  $b_l$  can be expressed as a sum over certain subgraphs. It is interesting  
21 to see whether  $A_l$ ,  $\lambda = 0$ , structural properties of a graph.

22 We may call (38) a parametric representation of  $H_c$ . In computation, the  
23 parameter  $\lambda_i$  plays very important roles. The choice of the parameter usually  
24 depends on the properties of the given graph. For a complete graph  $K_n$ , let  
25  $\lambda_i = 1$ ,  $i = 1, \dots, n$ . It follows from (39) that

$$\begin{array}{l} \text{26} \\ \text{27} \\ \text{28} \\ \text{29} \end{array} \quad (40) \quad A_l^{(1)} = \begin{cases} n!, & \text{if } l = 1 \\ 0, & \text{otherwise.} \end{cases}$$

30 By (38)

$$\begin{array}{l} \text{31} \\ \text{32} \\ \text{33} \end{array} \quad (41) \quad H_c = \frac{1}{2}(n-1)!.$$

34 For a complete bipartite graph  $K_{n_1 n_2}$ , let  $\lambda_i = 0$ ,  $i = 1, \dots, n$ . By (39),

$$\begin{array}{l} \text{35} \\ \text{36} \\ \text{37} \\ \text{38} \end{array} \quad (42) \quad A_l = \begin{cases} -n_1! n_2! \delta_{n_1 n_2}, & \text{if } l = 2 \\ 0, & \text{otherwise.} \end{cases}$$

39 Theorem 7.2 leads to

$$\begin{array}{l} \text{40} \\ \text{41} \\ \text{42} \end{array} \quad (43) \quad H_c = \frac{1}{n_1 + n_2} n_1! n_2! \delta_{n_1 n_2}.$$

Now, we consider an asymmetrical approach. Theorem 3.3 leads to

$$(44) \quad \det \mathbf{K}(t = 1, t_1, \dots, t_n; l|l) \\ = \sum_{I \subseteq \mathbf{n} - \{l\}} (-1)^{|I|} \prod_{i \in I} t_i \prod_{j \in I} (D_j + \lambda_j t_j) \det \mathbf{A}^{(\lambda)}(\bar{I} \cup \{l\} | \bar{I} \cup \{l\}).$$

By (3) and (16) we have the following asymmetrical result:

THEOREM 7.3.

$$(45) \quad H_c = \frac{1}{2} \sum_{I \subseteq \mathbf{n} - \{l\}} (-1)^{|I|} \text{per } \mathbf{A}^{(\lambda)}(I|I) \det \mathbf{A}^{(\lambda)}(\bar{I} \cup \{l\} | \bar{I} \cup \{l\})$$

which reduces to Goulden–Jackson’s formula when  $\lambda_i = 0, i = 1, \dots, n$  [MM64].

## 8. Various font features of the amsmath package

8.1. *Bold versions of special symbols.* In the amsmath package `\boldsymbol` is used for getting individual bold math symbols and bold Greek letters—everything in math except for letters of the Latin alphabet, where you’d use `\mathbf`. For example,

```
A_\infty + \pi A_0 \sim
\mathbf{A}_\infty + \pi \mathbf{A}_0
\mathbf{\pi} \mathbf{A}_0
```

looks like this:

$$A_\infty + \pi A_0 \sim \mathbf{A}_\infty + \pi \mathbf{A}_0$$

8.2. *“Poor man’s bold”.* If a bold version of a particular symbol doesn’t exist in the available fonts, then `\boldsymbol` can’t be used to make that symbol bold. At the present time, this means that `\boldsymbol` can’t be used with symbols from the `msam` and `msbm` fonts, among others. In some cases, poor man’s bold (`\pmb`) can be used instead of `\boldsymbol`:

$$\frac{\partial x}{\partial y} \bigg| \frac{\partial y}{\partial z}$$

```
[\frac{\partial x}{\partial y} \bigg| \frac{\partial y}{\partial z}]
\pmb{\bigg|}
```

So-called “large operator” symbols such as  $\sum$  and  $\prod$  require an additional command, `\mathop`, to produce proper spacing and limits when `\pmb` is used. For further details see *The T<sub>E</sub>Xbook*.

$$\sum_{\substack{i < B \\ i \text{ odd}}} \prod_{\kappa} \kappa F(r_i) \quad \sum_{\substack{i < B \\ i \text{ odd}}} \prod_{\kappa} \kappa(r_i)$$

```

1  \[\sum_{\substack{i<B\\\text{\$i\$ odd}}}]
2  \prod_{\kappa} \kappa F(r_i)\quad
3  \mathop{\pmb{\sum}}_{\substack{i<B\\\text{\$i\$ odd}}}
4  \mathop{\pmb{\prod}}_{\kappa} \kappa(r_i)
5  \]

```

## 9. Compound symbols and other features

9.1. *Multiple integral signs.* `\iint`, `\iiint`, and `\iiiiint` give multiple integral signs with the spacing between them nicely adjusted, in both text and display style. `\idotsint` gives two integral signs with dots between them.

$$(46) \quad \iint_A f(x, y) dx dy \quad \iiint_A f(x, y, z) dx dy dz$$

$$(47) \quad \idotsint_A f(w, x, y, z) dw dx dy dz \quad \int \cdots \int_A f(x_1, \dots, x_k)$$

9.2. *Over and under arrows.* Some extra over and under arrow operations are provided in the `amsmath` package. (Basic L<sup>A</sup>T<sub>E</sub>X provides `\overrightarrow` and `\overleftarrow`).

```

21  \overrightarrow{\psi_{\delta}(t) E_t h} = \psi_{\delta}(t) E_t h
22  \overleftarrow{\psi_{\delta}(t) E_t h} = \psi_{\delta}(t) E_t h
23  \overleftarrow{\overrightarrow{\psi_{\delta}(t) E_t h}} = \psi_{\delta}(t) E_t h
24  \overrightarrow{\overleftarrow{\psi_{\delta}(t) E_t h}} = \psi_{\delta}(t) E_t h
25
26  \begin{align*}
27  \overrightarrow{\psi_{\delta}(t) E_t h}& \\
28  =\underrightarrow{\psi_{\delta}(t) E_t h}& \\
29  \overleftarrow{\psi_{\delta}(t) E_t h}& \\
30  =\underleftarrow{\psi_{\delta}(t) E_t h}& \\
31  \overleftarrow{\overrightarrow{\psi_{\delta}(t) E_t h}}& \\
32  =\underleftarrow{\overrightarrow{\psi_{\delta}(t) E_t h}}& \\
33  \end{align*}

```

These all scale properly in subscript sizes:

$$\int_{\overrightarrow{AB}} ax dx$$

```

38  \[\int_{\overrightarrow{AB}} ax\,dx\]

```

9.3. *Dots.* Normally you need only type `\dots` for ellipsis dots in a math formula. The main exception is when the dots fall at the end of the formula; then you need to specify one of `\dotsc` (series dots, after a comma), `\dotscb`

1 (binary dots, for binary relations or operators), `\dotsm` (multiplication dots),  
2 or `\dotsi` (dots after an integral). For example, the input

3 Then we have the series  $A_1, A_2, \dots$ ,  
4 the regional sum  $A_1 + A_2 + \dots$ ,  
5 the orthogonal product  $A_1 A_2 \dots$ ,  
6 and the infinite integral  
7  $\int \int \dots$ .

8 produces

10 Then we have the series  $A_1, A_2, \dots$ , the regional sum  $A_1 + A_2 +$   
11  $\dots$ , the orthogonal product  $A_1 A_2 \dots$ , and the infinite integral

$$\int_{A_1} \int_{A_2} \dots$$

14 9.4. *Accents in math.* Double accents:

15  $\hat{H} \check{C} \tilde{T} \acute{A} \grave{G} \dot{D} \ddot{D} \breve{B} \bar{B} \vec{V}$

17  $\hat{\hat{H}}$   $\check{\check{C}}$   $\tilde{\tilde{T}}$   $\acute{\acute{A}}$   $\grave{\grave{G}}$   $\dot{\dot{D}}$   $\ddot{\ddot{D}}$   $\breve{\breve{B}}$   $\bar{\bar{B}}$   $\vec{\vec{V}}$   
18  $\hat{\check{H}}$   $\check{\acute{C}}$   $\tilde{\grave{T}}$   $\acute{\grave{A}}$   $\grave{\dot{G}}$   $\dot{\ddot{D}}$   $\ddot{\breve{D}}$   $\breve{\bar{B}}$   $\bar{\vec{B}}$   $\vec{\vec{V}}$   
19  $\hat{\tilde{H}}$   $\check{\tilde{C}}$   $\tilde{\acute{T}}$   $\acute{\acute{A}}$   $\grave{\grave{G}}$   $\dot{\dot{D}}$   $\ddot{\ddot{D}}$   $\breve{\breve{B}}$   $\bar{\bar{B}}$   $\vec{\vec{V}}$   
20  $\hat{\grave{H}}$   $\check{\grave{C}}$   $\tilde{\tilde{T}}$   $\acute{\acute{A}}$   $\grave{\grave{G}}$   $\dot{\dot{D}}$   $\ddot{\ddot{D}}$   $\breve{\breve{B}}$   $\bar{\bar{B}}$   $\vec{\vec{V}}$   
21  $\hat{\dot{H}}$   $\check{\dot{C}}$   $\tilde{\dot{T}}$   $\acute{\acute{A}}$   $\grave{\grave{G}}$   $\dot{\dot{D}}$   $\ddot{\ddot{D}}$   $\breve{\breve{B}}$   $\bar{\bar{B}}$   $\vec{\vec{V}}$   
22  $\hat{\ddot{H}}$   $\check{\ddot{C}}$   $\tilde{\ddot{T}}$   $\acute{\acute{A}}$   $\grave{\grave{G}}$   $\dot{\dot{D}}$   $\ddot{\ddot{D}}$   $\breve{\breve{B}}$   $\bar{\bar{B}}$   $\vec{\vec{V}}$

23 This double accent operation is complicated and tends to slow down the pro-  
24 cessing of a  $\LaTeX$  file.

25 9.5. *Dot accents.* `\dddots` and `\ddddots` are available to produce triple and  
26 quadruple dot accents in addition to the `\dot` and `\ddot` accents already avail-  
27 able in  $\LaTeX$ :

28  $\dddot{Q} \quad \ddddot{R}$

29  $\int \int \int \int \dots$

31 9.6. *Roots.* In the `amsmath` package `\leftroot` and `\uproot` allow you to  
32 adjust the position of the root index of a radical:

33  $\sqrt[\leftroot{-2}\uproot{2}]{\beta k}$

34 gives good positioning of the  $\beta$ :

35  $\sqrt[\beta]{k}$

37 9.7. *Boxed formulas.* The command `\boxed` puts a box around its argu-  
38 ment, like `\fbox` except that the contents are in math mode:

39  $\boxed{W_t - F \subseteq V(P_i) \subseteq W_t}$

40  $\boxed{W_t - F \subseteq V(P_i) \subseteq W_t}$

41

42

1 9.8. *Extensible arrows.* `\xleftarrow` and `\xrightarrow` produce arrows  
 2 that extend automatically to accommodate unusually wide subscripts or su-  
 3 perscripts. The text of the subscript or superscript are given as an optional  
 4 resp. mandatory argument: Example:

$$5 \quad 0 \underset{\zeta}{\xleftarrow{\alpha}} F \times \triangle[n-1] \xrightarrow{\partial_0 \alpha(b)} E^{\partial_0 b}$$

6  
 7  
 8 `\[0 \xleftarrow[\zeta]{\alpha} F \times \triangle[n-1]`  
 9 `\xrightarrow{\partial_0 \alpha(b)} E^{\partial_0 b}\]`

10 9.9. `\overset`, `\underset`, and `\sideset`. Examples:

$$11 \quad \overset{*}{X} \quad X \quad \underset{*}{X}$$

12  
 13  
 14 `\[\overset{*}{X} \qquad \underset{*}{X} \qquad \]`  
 15 `\overset{a}{\underset{b}{X}}\]`

16 The command `\sideset` is for a rather special purpose: putting symbols  
 17 at the subscript and superscript corners of a large operator symbol such as  $\sum$   
 18 or  $\prod$ , without affecting the placement of limits. Examples:

$$19 \quad \overset{*}{\underset{*}{\prod}_k} \quad \sum'_{0 \leq i \leq m} E_i \beta x$$

20  
 21  
 22 `\[\sideset{_*^*}{_*^*}\prod_k \qquad`  
 23 `\sideset{}{\prime}\sum_{0 \leq i \leq m} E_i \beta x`  
 24 `\]`

25 9.10. *The `\text` command.* The main use of the command `\text` is for  
 26 words or phrases in a display:

$$27 \quad \mathbf{y} = \mathbf{y}' \quad \text{if and only if} \quad y'_k = \delta_k y_{\tau(k)}$$

28  
 29  
 30 `\[\mathbf{y} = \mathbf{y}' \quad \text{if and only if} \quad`  
 31 `y'_k = \delta_k y_{\tau(k)}\]`

32 9.11. *Operator names.* The more common math functions such as `\log`, `\sin`,  
 33 and `\lim` have predefined control sequences: `\log`, `\sin`, `\lim`. The `amsmath`  
 34 package provides `\DeclareMathOperator` and `\DeclareMathOperator*` for  
 35 producing new function names that will have the same typographical treat-  
 36 ment. Examples:

$$37 \quad \|f\|_{\infty} = \text{ess sup}_{x \in R^n} |f(x)|$$

38  
 39 `\[\norm{f}_{\infty} =`  
 40 `\text{esssup}_{x \in R^n} \abs{f(x)}\]`

41  $\text{meas}_1 \{u \in R_+^1 : f^*(u) > \alpha\} = \text{meas}_n \{x \in R^n : |f(x)| \geq \alpha\} \quad \forall \alpha > 0.$

42

$\frac{1}{2}$  `\[\meas_1\{u\in R_{+}^1\colon f^*(u)>\alpha\}`  
 $\frac{2}{2}$  `=\meas_n\{x\in R^n\colon \abs{f(x)}\geq\alpha\}`  
 $\frac{3}{2}$  `\quad \forall\alpha>0.\]`  
 $\frac{4}{2}$  `\esssup` and `\meas` would be defined in the document preamble as  
 $\frac{5}{2}$   
 $\frac{6}{2}$  `\DeclareMathOperator*\{\esssup}\{ess\,sup\}`  
 $\frac{7}{2}$  `\DeclareMathOperator\{\meas}\{meas\}`

$\frac{8}{2}$  The following special operator names are predefined in the `amsmath` pack-  
 $\frac{9}{2}$  age: `\varlimsup`, `\varliminf`, `\varinjlim`, and `\varprojlim`. Here's what  
 $\frac{10}{2}$  they look like in use:

$$\frac{11}{2} \quad (48) \quad \overline{\lim}_{n \rightarrow \infty} Q(u_n, u_n - u^\#) \leq 0$$

$$\frac{13}{2} \quad (49) \quad \underline{\lim}_{n \rightarrow \infty} |a_{n+1}| / |a_n| = 0$$

$$\frac{15}{2} \quad (50) \quad \varinjlim (m_i^\lambda)^* \leq 0$$

$$\frac{16}{2} \quad (51) \quad \varprojlim_{p \in S(A)} A_p \leq 0$$

$\frac{18}{2}$  `\begin{align}`  
 $\frac{19}{2}$  `&\varlimsup_{n \rightarrow \infty}`  
 $\frac{20}{2}$  `\mathcal{Q}(u_n, u_n - u^\#) \le 0 \\`  
 $\frac{21}{2}$  `&\varliminf_{n \rightarrow \infty}`  
 $\frac{22}{2}$  `\left\lvert a_{n+1} \right\rvert / \left\lvert a_n \right\rvert = 0 \\`  
 $\frac{23}{2}$  `&\varinjlim (m_i^\lambda)^* \le 0 \\`  
 $\frac{24}{2}$  `&\varprojlim_{p \in S(A)} A_p \le 0`  
 $\frac{25}{2}$  `\end{align}`

$\frac{27}{2}$  9.12. `\mod` and *its relatives*. The commands `\mod` and `\pod` are variants  
 $\frac{28}{2}$  of `\pmod` preferred by some authors; `\mod` omits the parentheses, whereas `\pod`  
 $\frac{29}{2}$  omits the ‘mod’ and retains the parentheses. Examples:

$$\frac{30}{2} \quad (52) \quad x \equiv y + 1 \pmod{m^2}$$

$$\frac{32}{2} \quad (53) \quad x \equiv y + 1 \mod m^2$$

$$\frac{33}{2} \quad (54) \quad x \equiv y + 1 \pod m^2$$

$\frac{35}{2}$  `\begin{align}`  
 $\frac{36}{2}$  `x \equiv y + 1 \pmod{m^2} \\`  
 $\frac{37}{2}$  `x \equiv y + 1 \mod{m^2} \\`  
 $\frac{38}{2}$  `x \equiv y + 1 \pod{m^2}`  
 $\frac{39}{2}$  `\end{align}`

$\frac{40}{2}$  9.13. *Fractions and related constructions*. The usual notation for binomi-  
 $\frac{41}{2}$  als is similar to the fraction concept, so it has a similar command `\binom` with  
 $\frac{42}{2}$

1 two arguments. Example:

$$\begin{aligned}
 & \sum_{\gamma \in \Gamma_C} I_\gamma = 2^k - \binom{k}{1} 2^{k-1} + \binom{k}{2} 2^{k-2} \\
 & + \dots + (-1)^l \binom{k}{l} 2^{k-l} + \dots + (-1)^k \\
 & = (2-1)^k = 1
 \end{aligned}$$

```

8 \begin{equation}
9 \begin{split}
10 [\sum_{\gamma \in \Gamma_C} I_\gamma &
11 = 2^k - \binom{k}{1} 2^{k-1} + \binom{k}{2} 2^{k-2} \\
12 & \& \quad + \dots + (-1)^l \binom{k}{l} 2^{k-l} \\
13 & + \dots + (-1)^k \\
14 & \& = (2-1)^k = 1 \\
15 \end{split}
16 \end{equation}

```

17 There are also abbreviations

```

19 \dfrac          \dbinom
20 \tfrac          \tbinom

```

21 for the commonly needed constructions

```

22 {\displaystyle\frac ... }   {\displaystyle\binom ... }
23 {\textstyle\frac ... }     {\textstyle\binom ... }

```

24 The generalized fraction command `\genfrac` provides full access to the  
25 six T<sub>E</sub>X fraction primitives:

$$\text{(56)} \quad \overline{\frac{n+1}{2}} \quad \overwithdelims: \left\langle \frac{n+1}{2} \right\rangle$$

$$\text{(57)} \quad \atop \frac{n+1}{2} \quad \atopwithdelims: \left( \frac{n+1}{2} \right)$$

$$\text{(58)} \quad \above \frac{n+1}{2} \quad \abovewithdelims: \left[ \frac{n+1}{2} \right]$$

```

33 \text{\cn{over}: }&\genfrac{}{}{}{}{n+1}{2}&
34 \text{\cn{overwithdelims}: }&
35 \genfrac{\langle \rangle}{\rangle}{}{}{n+1}{2} \\
36 \text{\cn{atop}: }&\genfrac{}{}{0pt}{}{n+1}{2}&
37 \text{\cn{atopwithdelims}: }&
38 \genfrac{()}{)}{}{}{n+1}{2} \\
39 \text{\cn{above}: }&\genfrac{}{}{1pt}{}{n+1}{2}&
40 \text{\cn{abovewithdelims}: }&
41 \genfrac{[]}{]}{}{}{n+1}{2}

```



1 9.14. *Continued fractions.* The continued fraction

2

3 (59) 
$$\frac{1}{\sqrt{2} + \frac{1}{\sqrt{2} + \frac{1}{\sqrt{2} + \frac{1}{\sqrt{2} + \dots}}}}$$

4

5

6

7

8

9

10 can be obtained by typing

11 `\cfrac{1}{\sqrt{2}+`  
12 `\cfrac{1}{\sqrt{2}+`  
13 `\cfrac{1}{\sqrt{2}+`  
14 `\cfrac{1}{\sqrt{2}+`  
15 `\cfrac{1}{\sqrt{2}+\dotsb`  
16 `}}}}}`

17 Left or right placement of any of the numerators is accomplished by using  
18 `\cfrac[l]{}` or `\cfrac[r]{}` instead of `\cfrac`.

19

20 9.15. *Smash.* In `amsmath` there are optional arguments `t` and `b` for the  
21 plain `TEX` command `\smash`, because sometimes it is advantageous to be able  
22 to ‘smash’ only the top or only the bottom of something while retaining the  
23 natural depth or height. In the formula  $X_j = (1/\sqrt{\lambda_j})X'_j$  `\smash[b]` has been  
24 used to limit the size of the radical symbol.

25 `\$X_j=(1/\sqrt{\smash[b]{\lambda_j}})X'_j\$,`

26 Without the use of `\smash[b]` the formula would have appeared thus:  $X_j =$   
27  $(1/\sqrt{\lambda_j})X'_j$ , with the radical extending to encompass the depth of the subscript  
28  $j$ .

29

30 9.16. *The ‘cases’ environment.* ‘Cases’ constructions like the following  
31 can be produced using the `cases` environment.

32

33 (60) 
$$P_{r-j} = \begin{cases} 0 & \text{if } r-j \text{ is odd,} \\ r!(-1)^{(r-j)/2} & \text{if } r-j \text{ is even.} \end{cases}$$

34

35 `\begin{equation} P_{r-j}=`  
36 `\begin{cases}`  
37 `0& \text{if } r-j \text{ is odd},\ \`  
38 `r!\,(-1)^{(r-j)/2}& \text{if } r-j \text{ is even}.`  
39 `\end{cases}`  
40 `\end{equation}`

41 Notice the use of `\text` and the embedded math.

42

1 9.17. *Matrix.* Here are samples of the matrix environments, `\matrix`,  
 2 `\pmatrix`, `\bmatrix`, `\Bmatrix`, `\vmatrix` and `\Vmatrix`:

3

4

5

6

7

8

9

10

11

12

13

14

15

16

17

18

19

20

21

22

23

24

25

26

27

28

29

30

31

32

33

34

35

36

37

38

39

40

41

42

$$(61) \quad \begin{matrix} \vartheta & \varrho \\ \varphi & \varpi \end{matrix} \quad \begin{pmatrix} \vartheta & \varrho \\ \varphi & \varpi \end{pmatrix} \quad \begin{bmatrix} \vartheta & \varrho \\ \varphi & \varpi \end{bmatrix} \quad \left\{ \begin{matrix} \vartheta & \varrho \\ \varphi & \varpi \end{matrix} \right\} \quad \left| \begin{matrix} \vartheta & \varrho \\ \varphi & \varpi \end{matrix} \right| \quad \left\| \begin{matrix} \vartheta & \varrho \\ \varphi & \varpi \end{matrix} \right\|$$

```

\begin{matrix}
\vartheta & \varrho \\
\varphi & \varpi
\end{matrix}
\begin{pmatrix}
\vartheta & \varrho \\
\varphi & \varpi
\end{pmatrix}
\begin{bmatrix}
\vartheta & \varrho \\
\varphi & \varpi
\end{bmatrix}
\left\{
\begin{matrix}
\vartheta & \varrho \\
\varphi & \varpi
\end{matrix}
\right\}
\left|
\begin{matrix}
\vartheta & \varrho \\
\varphi & \varpi
\end{matrix}
\right|
\left\|
\begin{matrix}
\vartheta & \varrho \\
\varphi & \varpi
\end{matrix}
\right\|

```

To produce a small matrix suitable for use in text, use the `smallmatrix` environment.

32

33

34

35

36

37

38

39

40

41

42

```

\begin{math}
\bigl( \begin{smallmatrix}
a & b \\
c & d
\end{smallmatrix} \bigr)
\end{math}

```

To show the effect of the matrix on the surrounding lines of a paragraph, we put it here:  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and follow it with enough text to ensure that there will be at least one full line below the matrix.

`\hdotsfor{number}` produces a row of dots in a matrix spanning the given number of columns:

$$W(\Phi) = \begin{vmatrix} \frac{\varphi}{(\varphi_1, \varepsilon_1)} & 0 & \dots & 0 \\ \frac{\varphi^{k_{n2}}}{(\varphi_2, \varepsilon_1)} & \frac{\varphi}{(\varphi_2, \varepsilon_2)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \frac{\varphi^{k_{n1}}}{(\varphi_n, \varepsilon_1)} & \frac{\varphi^{k_{n2}}}{(\varphi_n, \varepsilon_2)} & \dots & \frac{\varphi^{k_{nn-1}}}{(\varphi_n, \varepsilon_{n-1})} & \frac{\varphi}{(\varphi_n, \varepsilon_n)} \end{vmatrix}$$

```

10 \[W(\Phi)= \begin{Vmatrix}
11 \dfrac{\varphi}{(\varphi_1, \varepsilon_1)}&0&\dots&0\
12 \dfrac{\varphi^{k_{n2}}}{(\varphi_2, \varepsilon_1)}&{\dfrac{\varphi}{(\varphi_2, \varepsilon_2)}}&\dots&0\
13 \dfrac{\varphi}{(\varphi_2, \varepsilon_2)}&\dots&0\
14 \hdotsfor{5}\
15 \dfrac{\varphi^{k_{n1}}}{(\varphi_n, \varepsilon_1)}&\dots&\dots&\dots&\dfrac{\varphi}{(\varphi_n, \varepsilon_n)}\
16 \dfrac{\varphi^{k_{n2}}}{(\varphi_n, \varepsilon_2)}&\dots&\dots&\dots&\dots\
17 \dfrac{\varphi^{k_{n,n-1}}}{(\varphi_n, \varepsilon_{n-1})}&\dots&\dots&\dots&\dots\
18 \dfrac{\varphi}{(\varphi_n, \varepsilon_n)}\
19 \end{Vmatrix}\]
20

```

The spacing of the dots can be varied through use of a square-bracket option, for example, `\hdotsfor[1.5]{3}`. The number in square brackets will be used as a multiplier; the normal value is 1.

9.18. *The `\substack` command.* The `\substack` command can be used to produce a multiline subscript or superscript: for example

```
\sum_{\substack{0 \leq i \leq m \\ 0 < j < n}} P(i, j)
```

produces a two-line subscript underneath the sum:

$$(62) \quad \sum_{\substack{0 \leq i \leq m \\ 0 < j < n}} P(i, j)$$

A slightly more generalized form is the `subarray` environment which allows you to specify that each line should be left-aligned instead of centered, as here:

Maybe "... as below"?

$$(63) \quad \sum_{\subarray{0 \leq i \leq m \\ 0 < j < n}} P(i, j)$$

```

38 \sum_{\begin{subarray}{l}
39 0 \leq i \leq m \\
40 \end{subarray}}
41 P(i, j)
42

```

1 9.19. *Big-g-g delimiters.* Here are some big delimiters, first in `\normalsize`:

2  
3 
$$\left( \mathbf{E}_y \int_0^{t_\varepsilon} L_{x,y^x(s)} \varphi(x) ds \right)$$
  
4  
5 `\[\biggl(\mathbf{E}_{\mathbf{E}}_{\mathbf{y}}  
6 \int_0^{t_\varepsilon} L_{x,y^x(s)} \varphi(x) ds  
7 \biggr)  
8 \]`

9 and now in `\Large` size:

10  
11 
$$\left( \mathbf{E}_y \int_0^{t_\varepsilon} L_{x,y^x(s)} \varphi(x) ds \right)$$
  
12  
13  
14 `\{\Large`  
15 `\[\biggl(\mathbf{E}_{\mathbf{E}}_{\mathbf{y}}  
16 \int_0^{t_\varepsilon} L_{x,y^x(s)} \varphi(x) ds  
17 \biggr)  
18 \}\}`

## 19 20 21 **References**

- 22 [Arn89] V. I. ARNOLD, *Mathematical Methods of Classical Mechanics*, second  
23 ed., *Graduate Texts in Mathematics* **60**, Springer, New York, 1989.
- 24 [DH76] W. DIFFIE and E. HELLMAN, New directions in cryptography, *IEEE*  
25 *Transactions on Information Theory* **22** (1976), 644–654.
- 26 [Fre94] D. H. FREMLIN, Cichon’s diagram, presented at the Séminaire Initia-  
27 tion à l’Analyse, G. Choquet, M. Rogalski, J. Saint Raymond, at the  
 Université Pierre et Marie Curie, Paris, 23e année., 1983/194.
- 28 [Fre08] D. H. FREMLIN, *Topological Riesz Spaces and Measure Theory*, Cam-  
29 bridge University Press, 2008.
- 30 [GJ81] I. P. GOULDEN and D. M. JACKSON, The enumeration of directed closed  
31 Euler trails and directed Hamiltonian circuits by Langrangian methods,  
32 *European Journal of Combinatorics* **2** (1981), 131–212.
- 33 [dGWH<sup>+</sup>92] C. DE GROOT, D. WÜRTZ, M. HANF, R. PEIKERT, T. KOLLER, and  
34 K. H. HOFFMANN, Stochastic optimization—efficient algorithms to solve  
35 complex problems, in *System Modelling and Optimization, Proceedings of*  
36 *the Fifteenth IFIP Conference (Zürich)* (P. KALL, ed.), Springer-Verlag,  
37 1992, pp. 546–555.
- 38 [HP73] F. HARARY and E. M. PALMER, *Graphical Enumeration*, Academic  
 Press, 1973.
- 39 [ILL89] R. IMPAGLIAZZO, L. LEVIN, and M. LUBY, Pseudo-random generation  
40 from one-way functions, in *Proc. 21st STOC* (Seattle, WA, USA), ACM,  
41 New York, 1989, pp. 12–24.
- 42

- 1 [Knu94] D. E. KNUTH, *The T<sub>E</sub>Xbook*, with illustrations by Duane Bibby, *Computers & Typesetting A*, Addison-Wesley Publishing Company, Reading, MA, 1994.
- 2
- 3
- 4 [KMY87a] M. KOJIMA, S. MIZUNO, and A. YOSHISE, *A New Continuation Method for Complementarity Problems With Uniform p-Functions*, Tech. Report B-194, Tokyo Inst. of Technology, Dept. of Information Sciences, Tokyo, 1987.
- 5
- 6
- 7 [KMY87b] M. KOJIMA, S. MIZUNO, and A. YOSHISE, *A Polynomial-Time Algorithm For a Class of Linear Complementarity Problems*, Tech. Report B-193, Tokyo Inst. of Technology, Dept. of Information Sciences, Tokyo, 1987.
- 8
- 9
- 10
- 11 [LO74] H. W. LENSTRA, JR. and F. OORT, Simple abelian varieties having a prescribed formal isogeny type., *J. Pure Appl. Algebra* **4** (1974), 47–53. MR 0279.14009. Zbl 50:7163. [http://dx.doi.org/10.1016/0022-4049\(74\)90029-2](http://dx.doi.org/10.1016/0022-4049(74)90029-2). Available at <http://www.math.leidenuniv.nl/~hwl/PUBLICATIONS/1973a/art.pdf>.
- 12
- 13
- 14
- 15
- 16 [LC84] C. J. LIU and Y. CHOW, On operator and formal sum methods for graph enumeration problems, *SIAM Journal of Algorithms and Discrete Methods* **5** (1984), 384–438.
- 17
- 18
- 19 [MM64] M. MARCUS and H. MINC, A survey of matrix theory and matrix inequalities, *Complementary Series in Mathematics* **14** (1964), 21–48.
- 20
- 21 [Mic38] A. D. MICHAL, Differential calculus in linear topological spaces, *Proc. nat. Acad. Sci. USA* **24** (1938), 340–342. JFM 64.0366.02.
- 22
- 23 [Mic48] A. D. MICHAL, *Matrix and Tensor Calculus*, *GALCIT Aeronautical Series*, John Wiley & Sons, Inc.; Chapman & Hall, Ltd., New York; London, 1948.
- 24
- 25 [MO08] A. MINASYAN and D. OSIN, *Normal Automorphisms of Relatively Hyperbolic Groups*, 2008. arXiv 0809.2408.
- 26
- 27 [MYK88] S. MIZUNO, A. YOSHISE, and T. KIKUCHI, *Practical Polynomial Time Algorithms for Linear Complementarity Problems*, Tech. Report 13, Tokyo Inst. of Technology, Dept. of Industrial Engineering and Management, Tokyo, April 1988.
- 28
- 29
- 30
- 31 [MA87] R. D. MONTEIRO and I. ADLER, *Interior Path Following Primal-Dual Algorithms, Part II: Quadratic Programming*, Working paper, Dept. of Industrial Engineering and Operations Research, August 1987.
- 32
- 33 [Ste70] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, NJ, 1970.
- 34
- 35 [Ye87] Y. YE, *Interior Algorithms for Linear, Quadratic and Linearly Constrained Convex Programming*, Ph.D. thesis, Stanford Univ., Dept. of Engineering–Economic Systems, Palo Alto, CA, July 1987.
- 36
- 37
- 38 [Zarh92] YU. G. ZARHIN, Abelian varieties having a reduction of K3 type, *Duke Math J.* **65** (1992), 511–527. MR 1154181. Zbl 0774.14039.
- 39
- 40 [Zara] YU. G. ZARHIN, *Algebra and Cryptography*, Private Communication.
- 41 [Zarb] YU. G. ZARHIN, *On Abel Groups*, Private Communication.
- 42

1  
2  
3  
4  
5  
6  
7  
8  
9  
10  
11  
12  
13  
14  
15  
16  
17  
18  
19  
20  
21  
22  
23  
24  
25  
26  
27  
28  
29  
30  
31  
32  
33  
34  
35  
36  
37  
38  
39  
40  
41  
42

(Received: December 24, 2004)  
(Revised: April 12, 2006)

AMS, PROVIDENCE, RHODE ISLAND  
*E-mail:* [tech-support@ams.org](mailto:tech-support@ams.org)

GEORGE MASON UNIVERSITY, FAIRFAX, VIRGINIA  
*E-mail:* [borisv@lk.net](mailto:borisv@lk.net)  
<http://borisv.lk.net>