A uniformly charged dielectric gel having charge density $\rho_v = \rho_o$ and dielectric constant of $\epsilon_r = 2$ is enclosed inside a dielectric shell with dielectric constant of $\epsilon_r = 5$ as shown in the figure. The dielectric shell is surrounded by free space ($\epsilon_r = 1$). Determine E, Φ, D, P in all regions.

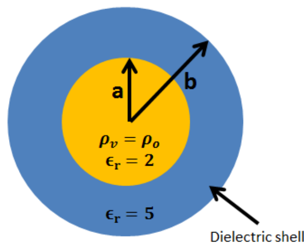

Figure 1: Given scenario.

In this problem we can consider there to be 3 regions, (1) $r < a$, (2) $a \leq r < b$, and (3) $r \geq b$. For finding the electric flux density Gauss's Law provides the most straight forward solution. Then, using the constitutive relation between electric flux density and electric field, $\vec{E}$ can be found. Thereafter $\vec{P}$ and Φ are directly related to $\vec{E}$.

Electric Flux Density:

$$\oint_S \vec{D} \cdot \hat{n} = Q_{en}$$
$$\int_0^{\pi} \int_0^{2\pi} D_r r^2 sin\theta d\phi d\theta = Q_{en}$$
$$4\pi r^2 D_r = Q_{en}$$

Region 1

$$Q_{en} = \int_0^{\pi} \int_0^{2\pi} \int_0^{r} \rho_o r^2 sin\theta dr d\phi d\theta$$
$$Q_{en} = \frac{4\pi \rho_o r^3}{3}$$
$$4\pi r^2 D_r = \frac{4\pi \rho_o r^3}{3}$$
$$\vec{D} = \frac{\rho_o r^3}{3r^2}$$
$$\vec{D} = \frac{\rho_o r}{3}\hat{r}\ C/m^2$$

Region 2 and 3

$$Q_{en} = \int_0^{\pi} \int_0^{2\pi} \int_0^{a} \rho_o r^2 sin\theta dr d\phi d\theta$$
$$Q_{en} = \frac{4\pi\rho_o a^3}{3}$$
$$4\pi r^2 D_r = \frac{4\pi\rho_o a^3}{3}$$
$$\vec{D} = \frac{\rho_o a^3}{3r^2}\hat{r}\ C/m^2$$

Electric Field:

$$\vec{D} = \epsilon\vec{E}$$
$$\vec{E} = \frac{1}{\epsilon}\vec{D}$$
$$\vec{E} = \frac{1}{\epsilon_o\epsilon_r}\vec{D}$$

Region 1

$$\vec{E} = \frac{1}{\epsilon_o\epsilon_r}\vec{D}$$
$$\vec{E} = \frac{1}{\epsilon_o\epsilon_r}\frac{\rho_o r}{3}$$
$$\vec{E} = \frac{1}{2\epsilon_o}\frac{\rho_o r}{3}\hat{r}\ V/m$$

Region 2

$$\vec{E} = \frac{1}{\epsilon_o\epsilon_r}\vec{D}$$
$$\vec{E} = \frac{1}{\epsilon_o\epsilon_r}\frac{\rho_o a^3}{3r^2}$$
$$\vec{E} = \frac{1}{5\epsilon_o}\frac{\rho_o a^3}{3r^2}\hat{r}\ V/m$$

Region 3

$$\vec{E} = \frac{1}{\epsilon_o\epsilon_r}\vec{D}$$
$$\vec{E} = \frac{1}{\epsilon_o\epsilon_r}\frac{\rho_o a^3}{3r^2}$$
$$\vec{E} = \frac{1}{\epsilon_o}\frac{\rho_o a^3}{3r^2}\hat{r}\ V/m$$

Electric Polarization:

$$\vec{P} = (\epsilon_r - 1)\epsilon_o\vec{E}$$

Region 1

$$\vec{P} = (\epsilon_r - 1)\epsilon_o \vec{E}$$
$$\vec{P} = (2-1)\cancel{\epsilon_o}\frac{1}{2\cancel{\epsilon_o}}\frac{\rho_o r}{3}$$
$$\vec{P} = \frac{1}{2}\frac{\rho_o r}{3}$$
$$\vec{P} = \frac{\rho_o r}{6}\hat{r}\ C/m^2$$

Region 2

$$\vec{P} = (\epsilon_r - 1)\epsilon_o \vec{E}$$
$$\vec{P} = (5-1)\cancel{\epsilon_o}\frac{1}{5\cancel{\epsilon_o}}\frac{\rho_o a^3}{3r^2}$$
$$\vec{P} = \frac{4}{5}\frac{\rho_o a^3}{3r^2}$$
$$\vec{P} = \frac{4\rho_o a^3}{15r^2}\hat{r}\ C/m^2$$

Region 3

$$\vec{P} = (\epsilon_r - 1)\epsilon_o \vec{E}$$
$$\vec{P} = (1-1)\epsilon_o \vec{E}$$
$$\vec{P} = 0\hat{r}\ C/m^2$$

<u>Electric Potential:</u>

$$\Phi = -\int_{\infty}^{r} \vec{E}\cdot dl$$

Region 3

$$\Phi = -\int_{\infty}^{r} \vec{E}_3\cdot dl$$
$$\Phi = -\int_{\infty}^{r} \frac{1}{\epsilon_o}\frac{\rho_o a^3}{3r^2}\hat{r}\cdot\hat{r}dr$$
$$\Phi = -\frac{\rho_o a^3}{3\epsilon_o}\int_{\infty}^{r}\frac{1}{r^2}dr$$
$$\Phi = -\frac{\rho_o a^3}{3\epsilon_o}\left[-\frac{1}{r}\right]_{\infty}^{r}$$
$$\Phi = \frac{\rho_o a^3}{3\epsilon_o r}$$
$$\Phi = \frac{\rho_o a^3}{3\epsilon_o r}\ V$$

Region 2

$$\Phi = -\int_{\infty}^{r} \vec{E} \cdot dl$$
$$\Phi = -\int_{\infty}^{b} \vec{E}_3 \cdot dl - \int_{b}^{r} \vec{E}_2 \cdot dl$$
$$\Phi = -\int_{\infty}^{b} \frac{1}{\epsilon_o}\frac{\rho_o a^3}{3r^2}\hat{r}\cdot\hat{r}dr - \int_{b}^{r} \frac{1}{5\epsilon_o}\frac{\rho_o a^3}{3r^2}\hat{r}\cdot\hat{r}dr$$
$$\Phi = -\frac{\rho_o a^3}{3\epsilon_o}\int_{\infty}^{b} \frac{1}{r^2}dr - \frac{\rho_o a^3}{15\epsilon_o}\int_{b}^{r} \frac{1}{r^2}dr$$
$$\Phi = -\frac{\rho_o a^3}{3\epsilon_o}\left[-\frac{1}{r}\right]_{\infty}^{b} - \frac{\rho_o a^3}{15\epsilon_o}\left[-\frac{1}{r}\right]_{b}^{r}$$
$$\Phi = \frac{\rho_o a^3}{3\epsilon_o}\frac{1}{b} + \frac{\rho_o a^3}{15\epsilon_o}\left[\frac{1}{r} - \frac{1}{b}\right]$$
$$\Phi = \frac{\rho_o a^3}{3\epsilon_o b} + \frac{\rho_o a^3}{15\epsilon_o r} - \frac{\rho_o a^3}{15\epsilon_o b}\ V$$

Region 1

$$\Phi = -\int_{\infty}^{r} \vec{E} \cdot dl$$
$$\Phi = -\int_{\infty}^{b} \vec{E}_3 \cdot dl - \int_{b}^{a} \vec{E}_2 \cdot dl - \int_{a}^{r} \vec{E}_1 \cdot dl$$
$$\Phi = -\int_{\infty}^{b} \frac{1}{\epsilon_o}\frac{\rho_o a^3}{3r^2}\hat{r}\cdot\hat{r}dr - \int_{b}^{a} \frac{1}{5\epsilon_o}\frac{\rho_o a^3}{3r^2}\hat{r}\cdot\hat{r}dr - \int_{a}^{r} \frac{1}{2\epsilon_o}\frac{\rho_o r}{3}\hat{r}\cdot\hat{r}dr$$
$$\Phi = -\frac{\rho_o a^3}{3\epsilon_o}\int_{\infty}^{b} \frac{1}{r^2}dr - \frac{\rho_o a^3}{15\epsilon_o}\int_{b}^{a} \frac{1}{r^2}dr - \frac{\rho_o}{6\epsilon_o}\int_{a}^{r} rdr$$
$$\Phi = -\frac{\rho_o a^3}{3\epsilon_o}\left[-\frac{1}{r}\right]_{\infty}^{b} - \frac{\rho_o a^3}{15\epsilon_o}\left[-\frac{1}{r}\right]_{b}^{a} - \frac{\rho_o}{6\epsilon_o}\left[\frac{r^2}{2}\right]_{a}^{r}$$
$$\Phi = \frac{\rho_o a^3}{3\epsilon_o}\frac{1}{b} + \frac{\rho_o a^3}{15\epsilon_o}\left[\frac{1}{a} - \frac{1}{b}\right] - \frac{\rho_o}{12\epsilon_o}\left[r^2 - a^2\right]$$
$$\Phi = \frac{\rho_o a^3}{3\epsilon_o b} + \frac{\rho_o a^3}{15\epsilon_o a} - \frac{\rho_o a^3}{15\epsilon_o b} - \frac{\rho_o r^2}{12\epsilon_o} + \frac{\rho_o a^2}{12\epsilon_o}$$
$$\Phi = \frac{\rho_o a^3}{3\epsilon_o b} + \frac{\rho_o a^2}{15\epsilon_o} - \frac{\rho_o a^3}{15\epsilon_o b} - \frac{\rho_o r^2}{12\epsilon_o} + \frac{\rho_o a^2}{12\epsilon_o}$$
$$\Phi = \frac{\rho_o a^2}{\epsilon_o}\left[\frac{a}{3b} + \frac{1}{15} + \frac{1}{12}\right] - \frac{\rho_o}{\epsilon_o}\left[\frac{a^3}{15b} + \frac{r^2}{12}\right]\ V$$

Final Answer Summary:

	Region 1 ($r < a$)	Region 2 ($a \leq r < b$)	Region 3 ($r \geq b$)
$\vec{D}$ (C/m^2)	$\frac{\rho_o r}{3}\hat{r}$	$\frac{\rho_o a^3}{3r^2}\hat{r}$	$\frac{\rho_o a^3}{3r^2}\hat{r}$
$\vec{E}$ (V/m)	$\frac{\rho_o r}{6\epsilon_o}\hat{r}$	$\frac{1}{5\epsilon_o}\frac{\rho_o a^3}{3r^2}\hat{r}$	$\frac{1}{\epsilon_o}\frac{\rho_o a^3}{3r^2}\hat{r}$
$\vec{P}$ (C/m^2)	$\frac{\rho_o r}{6}\hat{r}$	$\frac{4\rho_o a^3}{15r^2}\hat{r}$	$0\hat{r}$
Φ (V)	$\frac{\rho_o a^2}{\epsilon_o}\left[\frac{a}{3b}+\frac{1}{15}+\frac{1}{12}\right]-\frac{\rho_o}{\epsilon_o}\left[\frac{a^3}{15b}+\frac{r^2}{12}\right]$	$\frac{\rho_o a^3}{3\epsilon_o b}+\frac{\rho_o a^3}{15\epsilon_o r}-\frac{\rho_o a^3}{15\epsilon_o b}$	$\frac{\rho_o a^3}{3\epsilon_o r}$